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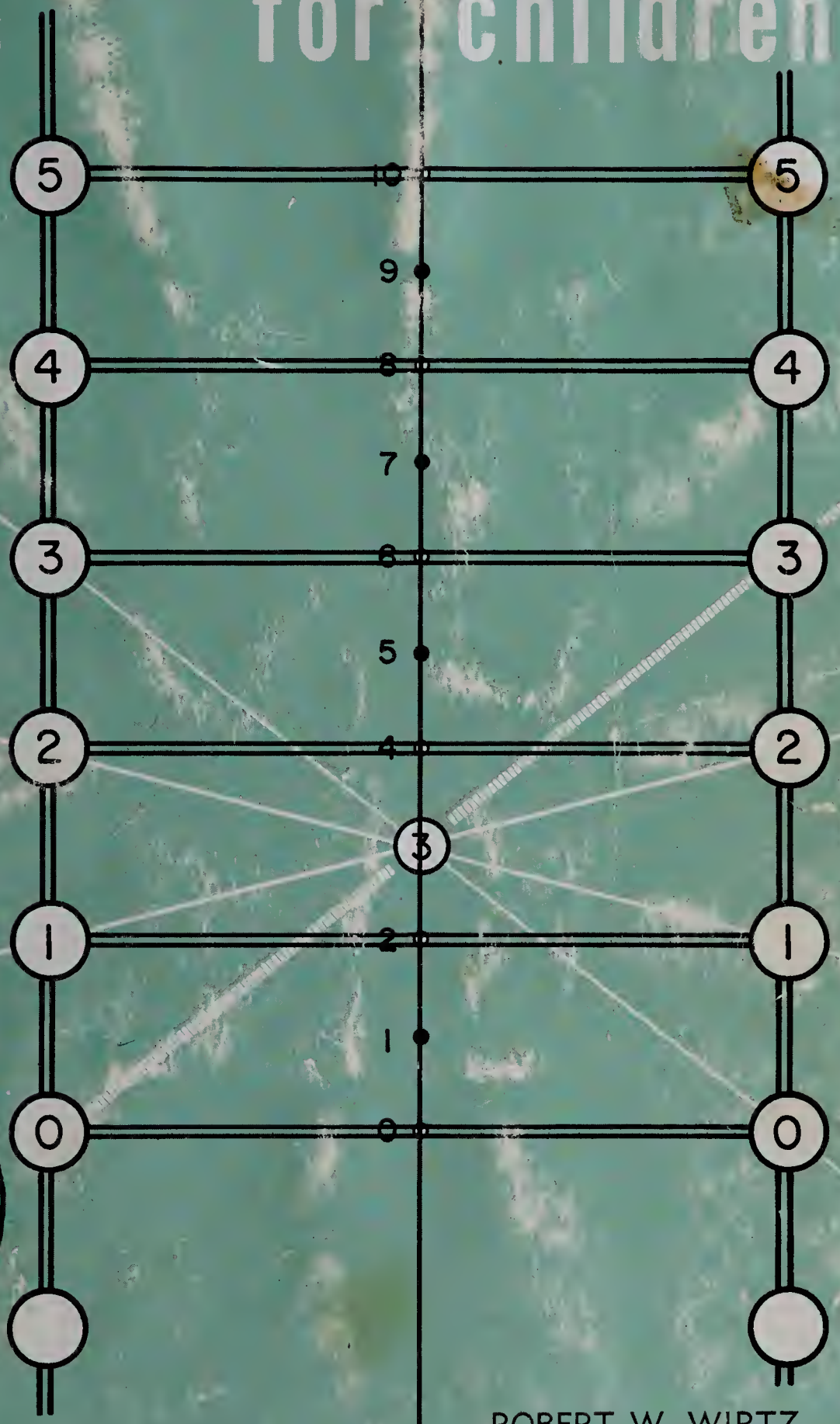

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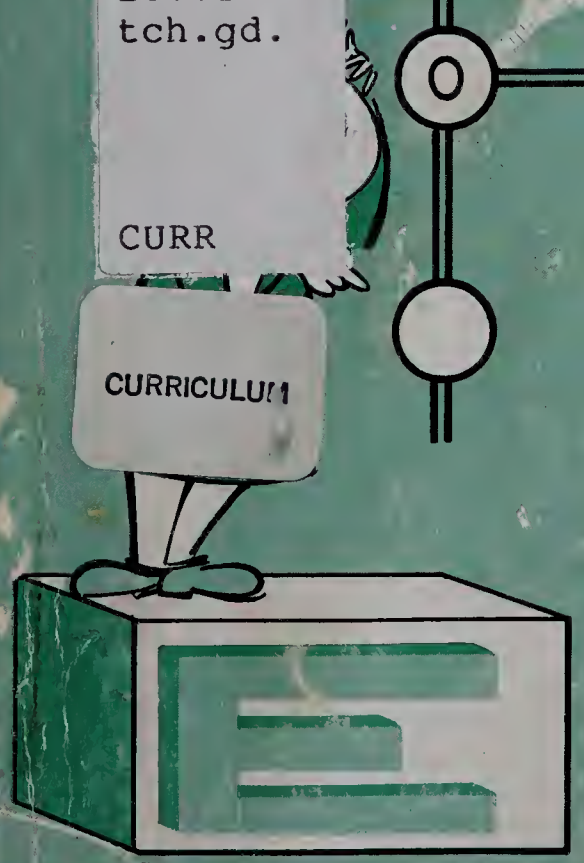

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LEVEL E
TEACHER'S GUIDE
MATH WORKSHOP FOR CHILDREN

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Preface

No previous work with *Math Workshop* materials is expected of children. *Level E*, as all levels of *Math Workshop* materials, has been prepared with the knowledge that many children will not have been introduced to anything other than traditional approaches to arithmetic. However, children who are already familiar with the *Math Workshop* will not find repetitions of previous work. Rather, unique activities have been designed that are basically new to all children.

Further, *Level E* starts with few assumptions of prior work in arithmetic. It moves steadily through all the content generally expected of children in about Grade 5 on standardized achievement tests.

Alec Marson's Machines and the Level E Program

Alec Marson is the featured character in *Level E*—yet he does not appear in it. He has entered several machines in the School Science Fair and disappeared, along with his supporting actress, his sister, Sue Marson.

The machines Alec left behind have no operating instructions. When the judges wanted to open them to look inside, they found there were no screws they could remove, no lids they could lift, no plates they could remove, and the metal was strange and impenetrable. What could the judges do?

Turn the dials, of course, and experiment. Find out what—if anything—the machines would do.

There is a pencil-like piece chained to the machine. "You write here" is written above the screen or plate. So, you write on the screen with the pencil-like piece.

What should you write? Write words, and there is no response from the machine.

Draw pictures—no response.

Write numerals—aha!—the machine goes into action.

You write 2; the machine writes 4. You write 0; the machine writes 0. You write 5; the machine writes 10. You write 1; the machine writes 2. You write 7 and you say to yourself, "I think the machine is going to write 14." It does. "And," you continue, "If I write 25, the machine will write 50 . . . I think!"

How can you find out? By experimenting. You look for a pattern in the responses of that machine. You recognize a certain relationship between what you write and what the machine writes.

Using the pattern you think you see, you predict what the machine is going to do. Then you test your prediction. If wrong, you look further for the relationship that ties all responses with what you write. If your prediction is correct, you predict and test again. Soon, you become quite confident that you can predict the machine's responses every time any number is written on the machine.

In the case outlined above, you might decide to call it a "doubling machine." If you think of it as such, your predictions are always proved to be correct.

Do you know what is inside the machine? No. You know absolutely nothing about what goes on inside that machine. You only see a relationship between the *input* (what you write) and the *output* (what the machine writes).

All you can say is that you have decided upon a rule that, when applied by you, leads to reliable predictions. Someone else may think of a different rule.

"The machine notes the number written on the input screen. It adds 4; doubles the result; and subtracts 8." And the proponent of this rule or explanation will make predictions that are as reliable as your doubling rule.

We will never know for sure what goes on inside the machine!

Our Purpose in Focusing on Alec Marson's Machines. The story line that unfolds in many ramifications throughout *Level E* has novelty to it, a strong flavor of the puzzle, and humorous developments. It also requires much computation and successful pattern-hunting to keep up with the story.

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Is it our purpose to sugar-coat drill work? No! (Though there is much needed "drill" in large piles along the way.) Instead, our rationale is that the story line parallels very closely mankind's efforts to understand nature; mankind's search for knowledge about the world he lives in. Nature's secrets are locked in just as the mechanisms of Alec's machines are locked in. All we can ever know about nature is that a certain input leads to a certain output, a certain stimulus produces a certain response, a certain cause is followed by a certain effect.

Where we see a relationship between cause and effect, we spin a theory—make an educated guess. Then we predict, and test our predictions. We keep on searching for a pattern that seems reliable. Our predictions are always proved to be correct. But we are never sure that we have unlocked nature's secret. We know that further down the road, we may find that our predictions are no longer reliable when one fails.

Suppose you are convinced that Alec's machine is a "doubling machine." You tabulate some results:

					↓	↓
You write	46	48	49	50	52	84
Machine writes	92	96	98	100	26	42

The results of those last two experiments were shockers! You didn't predict them. You must think back over your rule. "Perhaps," you argue, "the machine won't write any number larger than 100. So, it's a doubling machine until the input is greater than 50, and after that, it's a *halving machine!*"

So your predictions are again reliable—until another unexpected result turns up.

The scientific method. The story line about Alec Marson's Science Fair entries and the work of the judges to understand them has much in common with the method scientists use in advancing their understanding of the real world.

Scientists observe events, searching for a cause-effect relationship. Then, they try to find a pattern that will lead to formulation of a *rule* that can be used to predict what will happen in a given set of circumstances. They test their rule . . . predict, test . . . predict, test . . . until they are satisfied that their rule is reliable enough for the present.

Modern science has had shockers somewhat similar to the development imagined in which Alec's machine seemed to be a doubling machine up to a certain point, and then it seemed to be something quite different.

As scientists probed deeper and deeper into smaller and smaller objects—down into small groups of molecules, small groups of atoms, and into the atom itself—rules that had been completely reliable began, one by one, to fail as a reliable basis for prediction. Old rules had to be amended, patched up, revised, and hedged with exceptions until scientists finally came to the conclusion that rules that are reliable in the world of large things aren't usable in the sub-microscopic world. New laws of mechanics had to be formulated.

Other scientists found evidence that those same rules that are reliable in the world of large things—things around us—also break down when we probe into the far reaches of the universe. Einstein gave clear evidence that light seems to bend in the magnetic field of stars. Since our whole idea of straight lines is based on the way in which light travels, what happens to our rules that were reliable enough at a smaller scale of things?

The study of patterns. Mathematics is the study of patterns. It is the language that science uses to express patterns noticed in nature.

You can learn absolutely nothing about Alec Marson's machines if you do not see a pattern developing that relates input to output. You cannot easily express a pattern and use it for prediction without expressing it in some mathematical form.

Science is concerned with patterns in nature that can be expressed in some mathematical form. Those aspects of existence that do not lend themselves to such study are outside the realm of science and mathematics.

Pattern is a mechanism that makes memory possible. How can you pick out your friend's face from a thousand faces in an audience? Do you remember the color of his eyes, the contour of his left cheek, the proportion between the length and width of his head?

Impossible. You remember his face as a pattern in which all these and many other features are interrelated. You cannot remember enough detail; you can only remember the *pattern.*

Much agonizing is in store for anyone who feels he must commit all the details of arithmetic to memory—because no one can succeed at that impossible task. The joy of mathematics can be felt to the extent that we begin to see mathematics as a study of patterns rather than isolated facts and details.

Suppose that, after committing the 100 basic mul-

tiplication facts to memory, we set out to memorize the next 9,900 facts in which the factors are less than 100—all facts up to and including 99×99. We should certainly be overwhelmed by even the thought of it.

Patterns come to our rescue. When factors are both multiples of 10, such as 30×80, we rely on an observation we have made along the way. "All I need to remember is that $3 \times 8 = 24$ and $10 \times 10 = 100$ and $24 \times 100 = 2400$," or something on that order. The stickler will think $30 \times 80 = (3 \times 10) \times (8 \times 10) = (3 \times 8) \times (10 \times 10) = 24 \times 100 = 2400$.

Now, one also needs to know that $8 \times 40 = (8 \times 4) \times 10 = 320$. This pattern can be used whenever one factor is a multiple of 10.

Finally, a grand pattern that is so basic to accomplishing the test is illustrated by: $23 \times 39 = (20 + 3)\ (30 + 9) = (20 \times 30) + (20 \times 9) + (3 \times 30) + (3 \times 9)$. Since we know patterns that enable us quickly to find the product of each of those four indicated multiplications, only addition signs stand between the products. We are home free—provided, of course, that we can add.

The emergence of such grand patterns is an exciting reward for all the details that had to be handled along the way.

Each development in the study of Alec's machines leads toward the grasp of a pattern. Without a command of the basic facts of arithmetic, the pattern will be most elusive. So, a grasp of this basic vocabulary has its reward in the discovery of ways to anticipate or predict results of the machines.

Thus, the story line in *Level E* is fashioned to probe deep into the heart of mathematics.

Spiral curriculum

The concept of the spiral curriculum, as outlined by the *24th Yearbook of the National Council of Teachers of Mathematics*, has been used throughout all levels of the *Math Workshop*. At each level, from the first on, the main ideas of mathematics have been introduced. Each successive year's activities deal with these unifying ideas with increasing sophistication.

Less than mastery

However, children are not expected to "master" such elusive ideas as, say, multiplication of fractions or addition of positive and negative numbers—but they will meet all such ideas again and again, building always more layers of understanding.

The spiral curriculum rejects the "building-block" approach to arithmetic which has been used in the past to structure scope and sequence charts. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas, for repeated reconsideration at later times, continually nurturing the important concepts year after year as understanding develops.

The hope for "mastery" must be limited to that part of arithmetic that must be committed to memory—the vocabulary of arithmetic that must be instantly retrievable—the basic facts of arithmetic. Here, mastery can and must be achieved. But where big ideas are involved, mastery does not come in this way. It develops slowly and is based completely on understanding.

Approaches to individual activities

There are at least three generally successful ways in which individual pages or groups of pages can be handled in the classroom:

1) Use the *Work Text* to conclude a development;
2) use the page as the basis for discussion;
3) let children work independently.

1) In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching . . . with liberal use of manipulative material, the blackboard and class discussion. The *Work Text* is then used late in the development to tie down, or simply to record the progress that has been made to deeper understanding.

2) Other *Work Text* pages might, in contrast be gone through carefully, step-by-step, with the entire class—discussion encouraged.

3) A third approach might be called *Do-it-all-by-yourself*. Begin by saying, "Open your *Work Text* to (name the page). Look it over. Think about it. See if you can decide for yourself what is to be done. Wait a while before you ask for help. And let's make today a 'quiet day.' "

Now you feign reluctance to help but are, of course, watching carefully for those who haven't sufficient self-confidence to get started on their own. If several children are unable to get started, suggest that they work in pairs, helping each other quietly. Enlarge these helper combinations if necessary. Grouping like this will usually get the activity underway for everyone. Let time run out on the session if you wish, suggesting that those who haven't finished may want to take it home with them to finish.

Supplementary *Work Forms*

Many of the interesting and versatile page-forms developed in *Math Workshop* texts have been extracted and made available as Ditto and Multilith masters called *Work Forms*. In a few minutes, a teacher can convert one of these forms with pen or pencil into a rich supplementary activity for the whole class or for a group within the class. Providing, as they do, significant, independent learning activities for creative seatwork and meaningful homework, and taking into consideration, as they do, the wide range of individual abilities within any class, the *Work Forms* should prove very useful to busy teachers.

The 59 masters include:

10 Number Line Forms, numbered N1 to N10
14 Cross Number Forms, numbered C1 to C14
35 General Forms, numbered G1 to G35

The *Work Text* itself suggests many ways in which these forms can be used to help develop important ideas (and, at the same time, provide essential drill in the fundamental facts). Teachers can create similar activities by adding whatever numbers they want in a matter of minutes.

Countless other uses for the *Work Forms* can be found in the children's work texts and the teacher's guides to the other levels of the *Math Workshop*. The ideas in the particular adaptations found there are easily adaptable to any level of development.

Mastery Tests

The set of 59 *Work Forms* contains a battery of six tests, each of which presents all the basic facts of addition, subtraction, multiplication and division. An answer sheet is provided so that children can check their own or their neighbor's work. In addition, an individual record sheet enables each child to chart his own progress in terms of accuracy and time required to recall all the basic facts.

The *Work Forms*, then, provide the teacher with a flexibility and versatility never before available in an organized and integrated program. While they are not essential to the use of the *Math Workshop* texts, they are an invaluable supplement, an effective means of meeting individual needs, and a help to every teacher in expressing his own individual approach to helping children understand arithmetic.

Problem solving

Many people use the expressions "problem solving," "word problems," and "story problems" interchangeably in referring to such written questions as "Johnny lost 7 of his 11 pennies. How many has he left?"

Another example: "Farmer Brown had 8 posts. Each weighed 10 pounds. How much did the posts weigh altogether?" The answer, of course, is 80 pounds because $8 \times 10 = 80$ . . . or is it $10 \times 8 = 80$?

A slight change in the statement transforms it into a bit of a problem—into more than an example. "Farmer Brown had 8 posts. He set them out in a straight line, and 10 feet apart. How far is it between the end posts?"

To answer the question, you had better not look for "cue" words. Think for a moment; visualize or make a sketch of the situation. Once you understand the essence of the problem, then only is it reduced to an example from life of the pattern that "$7 \times 10 = 70$."

Now, both "examples" and "problems" have a vital role in helping children understand ways in which arithmetic applies in the physical world, but let us explore the use of "examples" in the teaching of elementary school arithmetic.

Professor W. W. Sawyer tells about a class of able fourth grade students that was asked: "Where would you ever use the fact that $24 \div 4 = 6$?" After considerable thought, a boy replied: "I might use it to answer some questions at school." "Yes," Professor Sawyer admitted, "but where else?" "Well," another student said after deep thought, "My mother might ask me to go to the store and get her $24 \div 4$ eggs."

"My mother never did that to me," was Professor Sawyer's reply.

Such episodes suggest a school program in which "examples" are not used as effective tools to help develop better understanding of arithmetic.

"Examples," when properly used, can be, so to say, the first baseman who clinches a very effective triple-play: from manipulative materials-to blackboard-to examples . . . examples from life.

The play begins in the world of objects children can feel and see and manipulate. A pattern is noted in otherwise unrelated events. Sketches and diagrams on the blackboard can help focus attention on the aspect of the experience that has been singled out for consideration. We need to be able to refer to a pattern we have noticed, so we attach a symbol or group of symbols as a name for the pat-

tern and thereby increase our command of the new language of arithmetic. This is an act of abstraction. We have given a name to a pattern.

Can each child use this new bit of language? Can he tell us about examples in which he recognizes this same pattern?

"Tell us a story. . . ." If the child can use his new language in this way, we've accomplished our immediate goal. If he can't, we must go back to the world of objects at hand and try again.

The child's response at the vital story-telling stage is the most reliable evidence a teacher can have that a satisfactory level of understanding has been reached. Without this immediately available evidence, the teacher could unwittingly leave some children lost in the strange world of abstract symbols—and that, for a child, is a chilling experience.

Children are brought to school by proud parents who tell us their Mary can count all the way to 25. But when she counts the fingers on her two hands, she ends up with the surprising conclusion that she has 13 fingers!

Other children are described as already knowing the addition facts. See for yourself: "Billie, how much is 4 plus 3?" He performs with an appropriate "7." But later we talk to Billie about a party where four children were sitting at a table, and then three more children arrived. "How many were there then?" He doesn't know.

So every good teacher knows he must find out what the language of arithmetic a child uses actually means to the child.

Most children, by this time, have learned that $4 \times 6 = 24$. They know that in 4 rows of checkers with 6 in a row, there are 24 checkers. If necessary, you can so arrange checkers or blocks or tongue depressors and carry out the counting process. And children would be sure that the same number of checkers could be arranged in 6 rows with 4 in each row. Further, they would realize that "4 columns with 6 in each" and "6 columns with 4 in each" are also descriptions of the arrangement. They would understand that "4 rows and 6 columns" and "6 rows and 4 columns" also refer to the same kinds of arrangements of 24 items of any kind.

This is the first part of the triple play . . . a recognition of a particular pattern in the world of things. Next, this pattern needs a name. A sensible name would be:

$$4 \times 6 = 24$$

This "naming" is the second part of the triple play. It has taken us out of the world of things and we are in the world of abstraction. We have written a series of symbols and have agreed that these symbols point clearly to the pattern we noted in the physical world.

But, let's hurry back to the more familiar world of things. "Think of that name for a pattern as a *headline.* Where might you encounter an example of that pattern?" Or, "Make up a story that fits that headline." Whether the stories are from the world of real things or the child's own world of imagination makes little difference—as long as those worlds operate in such a way that $4 \times 6 = 24$ in the same sense that an array of 4 rows and 6 columns of checkers contains 24 checkers.

"It's 6 blocks to school and I make two round-trips a day, so I walk 24 blocks." This story clearly fits the headline, even though 4 comes in only indirectly—because the child assumes that we understand that four one-way trips and two round-trips are the same distance.

"Twenty-four men went to the moon in four spaceships. Each ship has two pilots and a crew of four. There were six in each spaceship." The last sentence was added, we suppose, for those who want to know where 6 comes into the situation.

"There are 6 gallons of milk in 24 quarts." Does that story fit the headline? Does it point to a pattern that has something in common with that aspect of the array of checkers we named "$4 \times 6 = 24$?" Certainly. Imagine six gallon cartons of milk in a row. Now each carton is replaced by a column of four quart cartons. We now have 4 rows of cartons with 6 in a row—4 rows and 6 columns.

This is the third part of the triple play—finding a few of the unlimited number of "examples" from actual or imagined experiences which display the pattern we are discussing.

Thus, the search for examples is an essential part of the developmental work in the classroom. The children are the searchers. They are deepening their understanding of the relationships that exist between the abstract world of mathematics and the world we know through our senses, the physical world. This is a creative experience.

In such activities, "examples" play a fundamental role in the child's growth in comprehension of the meaning of mathematics.

In contrast, contrived "story problems" in text-

books, often used as tests, lead to a distortion of the relationship between mathematics and the real world—it leads children to say, "My mother might send me to the store for 24 ÷ 4 eggs."

Math Workshop work texts contain very few typical "word problems." However, in the teacher's guides, ample suggestions are given to help teachers stimulate all children in fruitful searches for examples of the patterns of arithmetic the children are considering.

Headlines and Fact Teams

We can think of no useful end served by arguing that one member of a fact team is a more appropriate "headline" for an example than are the other members of the team. Consider this team:

$4 \times 6 = 24$ $6 \times 4 = 24$ $24 \div 6 = 4$ $24 \div 4 = 6$

Our main emphasis here is that all these members are appropriate names for a pattern we see repeated over and over again in the real world. The focus is on the pattern, not on the particular expression that points to the pattern.

Arrange 24 checkers in 6 rows and 4 columns. That arrangement of things has something about it worth noticing. Any member of the fact team points to the pattern. No team member has any claim to being *most* appropriate.

The headlines, $24 \div 4 = 6$ and $4 \times 6 = 24$, are simply two different ways of pointing to a particular pattern. Some would argue that, "Six children confronted with the task of dividing 24¢ equally" ought to be analyzed in such fashion that "$24 \div \_\_\_\_ = 6$" is a proper statement of the problem, and "4¢" the proper solution. This centers the attention on discussion about "What is happening? What are the groups? Can you divide cents by people?"

The "understanding" in this approach is directed toward "understanding rules" to justify the argument that $24 \div 4 = 6$ is correct and that $24 \div 6 = 4$ in this instance is wrong. This must be terribly confusing to children and teachers—and parents.

Shift the concern to the pattern involved. What pattern must you recognize to help divide 24¢ equally among 6 children? The pattern we need to see is the same pattern we saw in the checkers or milk cartons. We named it with a fact team and each member of the team simply calls to mind the pattern!

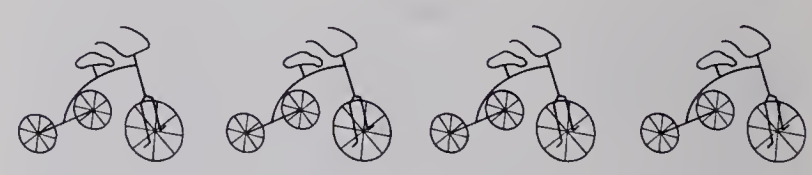
Suppose a child argues: "Half the children will get half the money—or 12¢ for 3 children. So, each child will get a third of 12¢—or 4¢ for each." So he suggests the headline "$(24 \div 2) \times 1/3 = 4$" as appropriate. Is he wrong? We hope not! . . . We hope that $(24 \div 2) \times 1/3 = 24 \div 6$.

Our concern is that children understand patterns and the relationships between the patterns of mathematics and the patterns of the world of experience. We hope they do not need rules to help them conclude that "each child gets 4¢."

Put the following incomplete headlines on the board. Ask the children to complete each so that it is a true statement and then make up a story that "fits the headline" in each case.

A.	$4 - 2 = \_\_\_\_$	B.	$3 \times 4 = \_\_\_\_$
C.	$\_\_\_\_ = 2 + 3$	D.	$12 = 4 + 4 + 4$
E.	$4 \times \_\_\_\_ = 12$	F.	$15 - 3 = \_\_\_\_$
G.	$\_\_\_\_ + 3 = 9$	H.	$\_\_\_\_ \div 4 = 3$
I.	$12 \div \_\_\_\_ = 4$	J.	$8 + 4 = \_\_\_\_$

Several of the headlines seem to point to the same pattern. Which ones? (Candidates will be B, D, E, H, I; and possibly an argument will be advanced for F.) How many fit the following sketch if we direct our attention to "wheels?"



"Finish all the proposed headlines."

$3 \times 4 = 12$ $4 \times 3 = 12$ $12 \quad 3 = 4$
$12 = 4 + 4 + 4$ $12 \quad 4 = 3$ $8 + 4 = 12$

Shall we accept all these headlines as appropriate? Does each point to a pattern that fits the real-life situation we are confronting? The easy way out would be to accept them all as different ways of saying something about the "number of wheels" in this case. That's the easy way . . . and why not take it?

Especially since the alternative has a gloomy outlook to it. We might soon be involved in a philosophical argument that would find us arguing that 3×4 does not equal 4×3. At least, we would have to spin a whole set of confusing rules so that, in light of those rules, we could say that one candidate's headline complies more fully than another. The result of such a contrived obstacle is to make "problem solving" a distasteful, complicated, lifeless effort to remember rules and recognize cue words that will help us recall from memory the right rules so we can come to a correct answer.

Any of the headlines we are considering is but another way of pointing to the pattern of the "number of wheels." As soon as a child has completed any one of the headlines that he thinks appropriate, he is in command of a fact of arithmetic that applies. How does he apply it? . . . if by "rules," he is apt to make a mistake; if by thinking about the situation, he will make a sensible reply.

But this part of the activity—recognizing that there are 12 wheels on the 4 tricycles—is only the initiating and less important part of a developmental situation.

"Now," we say, "think up other examples or stories that fit all those headlines." This leads us toward a creative activity in which children explore the relationship between arithmetic and the real world.

"Three chairs have 12 legs."

"There are 12 legs on 4 three-legged stools."

"In 3 rows of trees with 4 trees in a row, there are 12 trees."

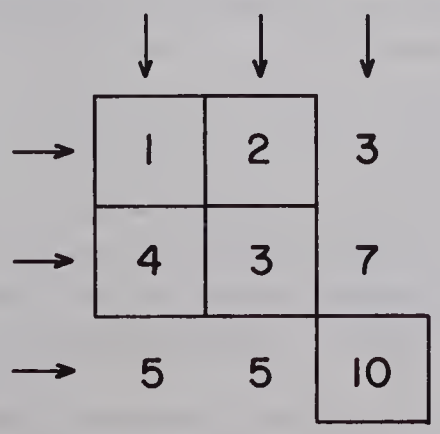
"On 3 wagons, there are 12 wheels."

"Two round-trips to Grandmother's is 12 miles, so she must live 3 miles away."

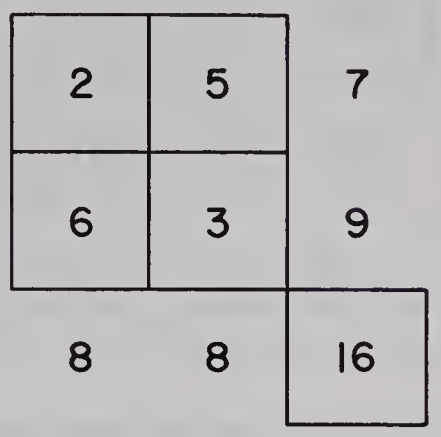
There is a common aspect about all these more or less seemingly diverse situations drawn from the real world. Any of the headlines above point to that common aspect. Any of the completed headlines contains the arithmetic each statement or example relies on.

We ought always to simplify the relationship between arithmetic and the physical world around us. We ought to worry when anyone tries to complicate and propose endless, useless, dangerous rules.

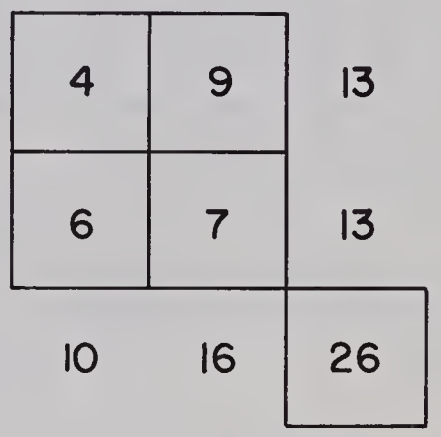
Below is a Cross Number Puzzle form. The problem is to select a number for each of the 4 cells so that if we add across each of the 2 rows and then add their sums, the grand total can serve equally well as the grand sum if we go on to add each of the two columns and total our results. Here is one group of 4 numbers that meets the test:



Perhaps you notice that the sum of each column is 5. Try another group of 4 numbers with the same pattern in the two columns—the sum is 8 in each.



Try an example in which the columns do not follow the pattern we notice:

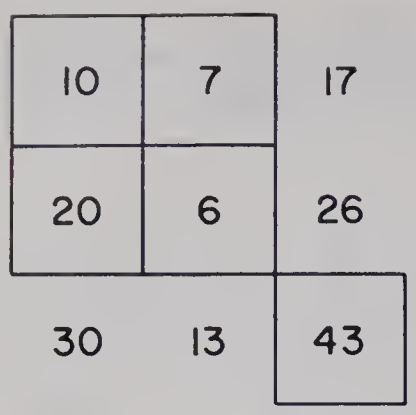


Now the rows add up to the same sum. There's a good bit of symmetry evident and some feel quite confident that this is a scheme that will keep on producing solutions to this puzzle. And it will. But the whole truth is much more startling.

We have explored this idea in the abstract manipulating numbers, inspecting sums, and sums of sums. But, bring it down to the world of things—cows, this time?

Farmer Brown put four cows in one field, nine in another, six in another and seven in the last field. In the two north fields, there are 13 cows and in the two south fields, there are 13 cows—or 26 in all.

Wouldn't it be surprising if there hadn't been the same number of cows regardless of the route over which he drove them? If we came out with any other result, we would think something unnatural had happened along the way. So, we are led to suspect that our puzzle will "work" regardless of what number of cows Mr. Brown puts in those 4 fields.



How clear the whole matter becomes when we take it out of the abstract and find a suitably similar example in the world of objects.

Incidentally, do you notice how quietly an example of "regrouping" and "carrying" was insinuated? If we want to add 17 cows and 26 cows, separate them into fields so that in one column of fields the numbers of cows are multiples of 10.

Headlines and "What's My Rule?"

Now, let's play "What's My Rule?" At first, teacher is "it." She asks the child to call out a number, such as 3. She then "applies her rule" and answers with "6." This example is recorded:

$3 \rightarrow 6$

As children call out numbers, and the teacher applies her rule, results accumulate:

$3 \rightarrow 6$
$5 \rightarrow 10$
$4 \rightarrow$

"I know what you are going to answer," Mary blurts out. "You are going to write 8."

Mary takes over from the teacher. The children keep calling out numbers, and Mary "applies the rule" in responding.

$4 \rightarrow 8$
$1 \rightarrow 2$
$2 \rightarrow$

"You're going to answer by saying 4!" By this time, many of the children think they know the rule.

"Now, let's use these results as a headline. What experience could we be talking about? We could be telling you how far we live from school, and then you tell us how many blocks in a round trip. If I say 2 because I live two blocks from school, you'll answer 4 because that's how far I would walk to school and back. Or maybe Mary and Dick put their books in two piles with the same number in each. The first number tells how big the piles were, and then you tell how many books they started with."

Or, "My father told me he would give me as much as I save from my allowance each week. If I save 4¢, he gives me 4¢ extra and I have 8¢ to put in the bank." And so forth, until the teacher tells the class she has another rule. The list of results accumulates like this:

$1 \rightarrow 8$
$5 \rightarrow 10$
$3 \rightarrow 8$
$0 \rightarrow$

"Five," Bill says firmly.

"If I say 2, you're going to say 7," someone else ventures.

When most children have indicated their grasp of the rule, the results become a headline. What story could fit that headline?

"A boy found a nickel. The first number tells how much he had to start with and the other number tells how much he had after he found the nickel."

"After we drove a ways, we had five miles left to go. If we had gone one mile, then we have five more to go and the whole trip is six miles."

In another "What's My Rule?" game, the teacher asks for a pair of numbers. She rejects some pairs because they don't fit her rule. The following headline is the result:

$3,1 \rightarrow 2$
$5,7 \rightarrow 6$
$6,2 \rightarrow 4$
$0,6 \rightarrow 3$
$4,2 \rightarrow$

"You're going to write 3 . . . and I know a story. "Bill had six books and his sister had two. They put them together and then divided them evenly—each got four."

"In my closet there were three shoes on the shelf and one on the floor. You know now that I have two pairs of shoes."

Another game produces this headline:

$5,7 \rightarrow 5$
$10,9 \rightarrow 9$
$8,5 \rightarrow 5$
$8,6 \rightarrow$

"You are going to answer '6' . . . and your rule is like this. Popsicles are 8¢ at one store and 6¢ at another. You can be sure I only spent 6¢."

So the game moves constantly back and forth

between abstractions of arithmetic and similar patterns in real-life situations. Before we leave each example, every child in the room (almost without exception) discovers the secret rule for himself, if not in the list of abstract numbers, then in the examples from life.

These simple games that involve all children included the idea of multiplication, addition, division, average and inequalities. At no time is any child asked to *verbalize* the "rule." If he could anticipate the result while the headline was being constructed, he was operating on a higher level of discovery than those who saw what was going on only as the examples from experience developed. In either case, the children were moving back and forth between the numbers of arithmetic and their use in the real world.

It is obvious that this game can deal with an unlimited variety of relationships on any level of maturity. Each child is challenged to participate on the highest level for which he is equipped by his own talent and experience.

The headline writer sometimes invents shorthand that seems strange, at first. The story writer needs some clues. Here is such a headline, and a group of hints:

$$\begin{array}{r} 1{,}2 \\ +0{,}3 \\ \hline , \end{array}$$

hints:

$$\begin{array}{r} 0{,}2 \\ +0{,}1 \\ \hline 0{,}3 \end{array} \qquad \begin{array}{r} 0{,}3 \\ +0{,}2 \\ \hline 1{,}0 \end{array} \qquad \begin{array}{r} 1{,}3 \\ +0{,}1 \\ \hline 1{,}4 \end{array} \qquad \begin{array}{r} 0{,}4 \\ +1{,}1 \\ \hline 2{,}0 \end{array} \qquad \begin{array}{r} 0{,}3 \\ 0{,}3 \\ \hline , \end{array}$$

Are there enough hints? Suppose there aren't, and the teacher must tell about the boy who first held up all the fingers on one hand and two on the other. Then he raised three more fingers. Now he held up all the fingers on two hands. So we complete the headline this way:

$$\begin{array}{r} 1{,}2 \\ +0{,}3 \\ \hline 2{,}0 \end{array}$$

Can you tell a version of the story that fits this headline? Probably, but again suppose we haven't much help from the class. So we say, "A girl had one nickel and two pennies. She earned three more pennies and wanted to change them so she had the fewest coins. She had one nickel and five pennies. She changed the five pennies into a nickel and then had only two coins—both nickels." The light begins to dawn. Let's try one of the hints.

$$\begin{array}{r} 0{,}3 \\ +0{,}2 \\ \hline 1{,}0 \end{array}$$

"Bill had three pennies in one pocket and two in another. At the bank he changed them for a nickel and had no pennies left."

"I had three fingers up and then put up two more. Then I had one handful of fingers up." But, someone argues, maybe he had three on one hand and two on the other. To fit the headline, we need a rule of some kind—like the "fewest coins" rule—perhaps we require that you must always fill out a handful of fingers if you can. Then the incomplete hint:

$$\begin{array}{r} 0{,}3 \\ +0{,}3 \\ \hline , \end{array}$$

Complete this headline and give an example that fits it.

"Three pennies and three pennies equals six pennies. Five of them can be changed to one nickel and there is one penny left or one nickel and one penny in all," (The headline 0,3 + 0,3 = 1, 1 can be read "zero comma three plus zero comma three equals one comma one.")

"We have three school days this week and three school days next week. That's the same as one week of school days and one more day."

$$\begin{array}{r} 1{,}0 \\ -0{,}3 \\ \hline , \end{array}$$

"I had a nickel and spent 3¢. That left me with no nickel and two pennies."

"I put down three out of a handful of fingers. There were only two left up."

$$\begin{array}{r} 2{,}2 \\ -\ 1{,}4 \\ \hline 0{,}3 \end{array}$$

A completed headline often stimulates a greater variety of responses—if such are encouraged.

"I have one nickel and four pennies left after spending 3¢. I started out with two nickels and two pennies."

"I can't put up three more fingers if I already have one handful up and four up on the other hand. I would be two fingers short."

$$\begin{array}{r} 1{,}2 \\ \times\ \ 3 \\ \hline , \end{array}$$

"Each of three girls had one nickel and two pennies. They put their money together and had three nickels and six pennies. They changed five of the pennies for a nickel and then had four nickels and one penny." And, eventually:

$$\begin{array}{r} 1,2 \\ 3\overline{)4,1} \end{array}$$

"Bill had three nickels and a penny. He spent a third of it. That is, one nickel and two pennies. Four nickels and one penny is the same as three nickels and six pennies. One third of each is one nickel and two pennies."

When the children have the idea worked out, a new headline is considered and obviously requires some new hints.

$$\begin{array}{rrrrrr} 0,8 & & 0,5 & 0,10 & 1,5 & 1,6 \\ +\ 0,4 & \text{hints:} & +\ 0,6 & +\ 0,\ 4 & +\ 0,5 & +\ 1,6 \\ \hline 1,0 & & 0,11 & 1,\ 2 & 1,10 & \end{array}$$

"There were eight eggs in one carton and four eggs in another. All could be put in one carton of one dozen—and none left over."

"The little hand of the clock has gone around to 8. In four more hours it will have made exactly one trip around the clock."

"A board 8 inches long and one 4 inches long can be cut out of a 1-foot board."

"My last birthday was eight months ago. My next birthday will be in four more months because eight months and four months are one year and no months."

Then there is the incomplete hint above—

$$\begin{array}{r} 1,6 \\ +\ 1,6 \\ \hline \end{array}$$

"A dozen and a half plus a dozen and a half is three dozen."

"Out of 3 feet of ribbon, I can cut two pieces, each 1 foot and 6 inches long."

All these headlines have a bit of a puzzle to them. The notation is strange, but it helps express clearly many events in the real world. Part of the purpose in providing children with such activities is exercise in using what some people call *denominate numbers*—adding 8 inches to 1 foot and 6 inches, and then changing 1 foot and 14 inches into 2 feet and 2 inches. But there is more.

Consider these headlines.

$$\begin{array}{r} 1,7 \\ +\ 0,6 \\ \hline \end{array} \qquad \begin{array}{rrrr} & 0,3 & 0,5 & 3,6 \\ \text{hints:} & +\ 0,7 & +\ 1,6 & +\ 1,6 \\ \hline & 1,0 & 2,1 & 5,2 \end{array}$$

"Jim had one dime and seven pennies. He earned six more pennies. He changed his dime and thirteen pennies into two dimes and three pennies."

$$\begin{array}{r} 3,2 \\ -\ 1,5 \\ \hline \end{array}$$

"Mary spent a dime for a candy bar and 5¢ for gum. She started out with three dimes and two pennies. She had one dime and seven pennies left."

$$\begin{array}{r} 1,4 \\ \times\ \ 3 \\ \hline \end{array}$$

"If each of three boys took a dime and four pennies to the store to make a purchase, they had three dimes and twelve pennies. First they changed their money into four dimes and two pennies."

One of the important dimensions to this activity is now apparent. In this final series of examples, we find that we no longer need the commas in the notation.

$$\begin{array}{r} 23 \\ -\ \ 8 \\ \hline \end{array}$$

"I had 23¢ and spent 8¢. I had one dime and five cents left, or 15¢."

On our somewhat circuitous route through hands and dozens and dimes, we have been exploring the idea of place value; and, we have been carrying and borrowing without staying long at any time in the abstract world of numbers. Every step we took was based solidly on the physical world of things.

This *headline* technique can be used to help develop every idea of arithmetic on every level of development. It's an old idea that is familiar to all teachers. All teachers have, from time to time, asked children to provide examples in which facts of arithmetic are reflected in life. But, this activity ought to become a theme permeating daily activity.

Examples called "story problems" have usually been reserved for use as tests of what children remember about trips through the abstract reaches of number manipulation. "Now that the class has learned a new group of division facts," we say, "let's see if they can use this new knowledge."

Children can be creators of their own examples—answers and all—and some of theirs will be much more difficult than any that could be included in a textbook. Since most of the creative activity depends on the spoken language, there is less of a reading problem to slow down the arithmetic. Reliance is on the speaking vocabulary which is usually several years ahead of the reading vocabulary.

Children are also given written examples. They must work out the appropriate headline, but that's only the beginning. As soon as they have found and correctly interpreted a headline, they are sent scurrying about for many other examples that would fit the same headline. In this way, usually trivial story problems launch a spirited search for suitable and more personally satisfying examples. Many children demonstrate remarkable ingenuity in creating a fascinating example to fit a headline that started from the question, "How many children are there in two sets of triplets?" You're apt to be out in space in no time at all, or deeply involved in a baseball game, with "two sets of outs" each "inning."

This activity obviously provides adequately for individual differences. There is opportunity for each child to participate on his own level—and to enjoy success.

Each pupil is urged to stretch his efforts in the direction of increased creativity.

Because the acitivities outlined here never lose intimate touch with the physical world, children can move on at an unexpected speed—and with complete understanding. Teachers become the thought-starters, the probers, the leaders who open to children the exciting world of mathematics. Once the interest in pattern and the search for its extension has begun, children are anxious to move on at their own speed. A breath of freedom and sense of adventure enters arithmetic.

If the method proposed here as a main classroom technique has been made to stand on its own feet, then the time has come to give away a secret. The secret was saved for the end—after all the arguments were in—so that skeptics would keep on reading.

The secret is a simple one and one easily demonstrated in any classroom. It is that children actually enjoy looking for their own examples in their own world of fact and fancy. Where this hunt for suitable illustrations has been used consistently in classrooms, it has never failed to excite keen interest in arithmetic. Children have good, old-fashioned fun as they deepen their mathematical understanding.

Other ideas from other levels

Many activities suggested for other levels of the *Math Workshop* are easily adaptable to many levels of maturity. Some are reviewed here as aids to creative classroom work that are rich in mathematical content, stimulating for all, and open-ended to provide for individual differences.

1. The language of addition and multiplication

In arithmetic, there are two processes that make it possible to *operate on* or think about a pair of numbers in order to suggest a particular third number. These basic processes are called addition and multiplication.

Addition or multiplication, and their inverses, subtraction or division, can be discussed in a few meaningful words. The key words used in addition and subtraction are: *two addends and a sum.* Addition and subtraction boil down to a search for a missing term: an addend or a sum, when two of the three terms are given.

	MISSING	GIVEN		
		Addend	Addend	Sum
2+3 = ____	sum	✔	✔	
1+____=8	addend	✔		✔
____+7 = 10	addend		✔	✔
7−3 = ____	addend		✔	✔
7−____ = 2	addend	✔		✔
____−10 = 15	sum	✔	✔	

The key words used in multiplication and division are: *two factors and a product.* Multiplication and division boil down to a search for a missing term: a factor or a product, when two of the three terms are given.

	MISSING	GIVEN		
		Factor	Factor	Product
3×4 = ____	product	✔	✔	
____×7 = 21	factor		✔	✔
5×____ = 30	factor	✔		✔
24÷3 = ____	factor		✔	✔
18÷____ = 9	factor		✔	✔
____÷6 = 7	product		✔	

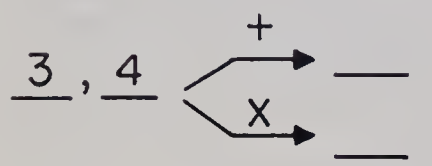
Main credit for this greatly simplified language belongs to the School Mathematics Study Group (SMSG). Instead of preserving the confusing vocabu-

lary that has been traditionally repeated in elementary arithmetic textbooks, SMSG has chosen to reflect the way mathematicians actually think and work. Looking at the operations of arithmetic in this way makes it possible for a pupil to have confidence enough to say, *If I can add, of course I can subtract, and if I can multiply, of course I can divide.*

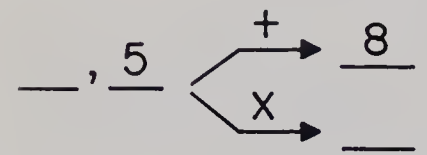
2. Basic fact practice games

When a pair of numbers is considered as addends and as factors that point to a sum and a product, four numbers are involved—the pair, a sum, and a product. *If any two are given, can you find the other two?* (Yes, unless 0 is given as one of the pair and as the product.)

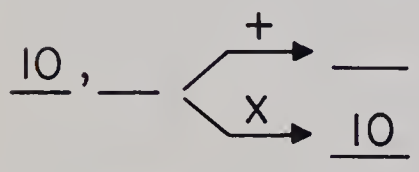
There are many ways to ask these questions. Three suggestions are given below:



(Answer: Sum must be 7 and product must be 12)



(Answer: Missing member of pair is 3 and missing product must be 15)



(Answer: Missing member of pair must be 1 and missing sum must be 11)

In another form, the questions may be asked this way:

	Product	Pair	Sum
A		3,2	
B	16		10
C	13	1,	
D		3,	6
E	7		

(Missing product is 6 and missing sum is 5)

(Missing pair must be 2,8 or 8,2)

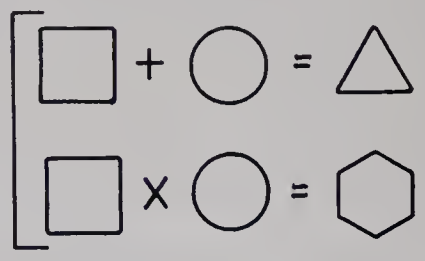
(Missing member of pair must be 13 and sum must be 14)

(Missing member of pair must be 3 and product must be 9)

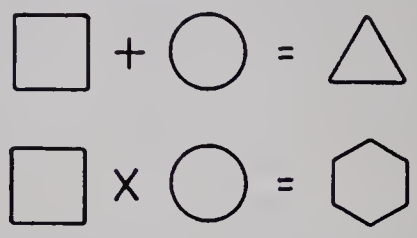
(Missing pair must be 1,7 or 7,1 and sum must be 8)

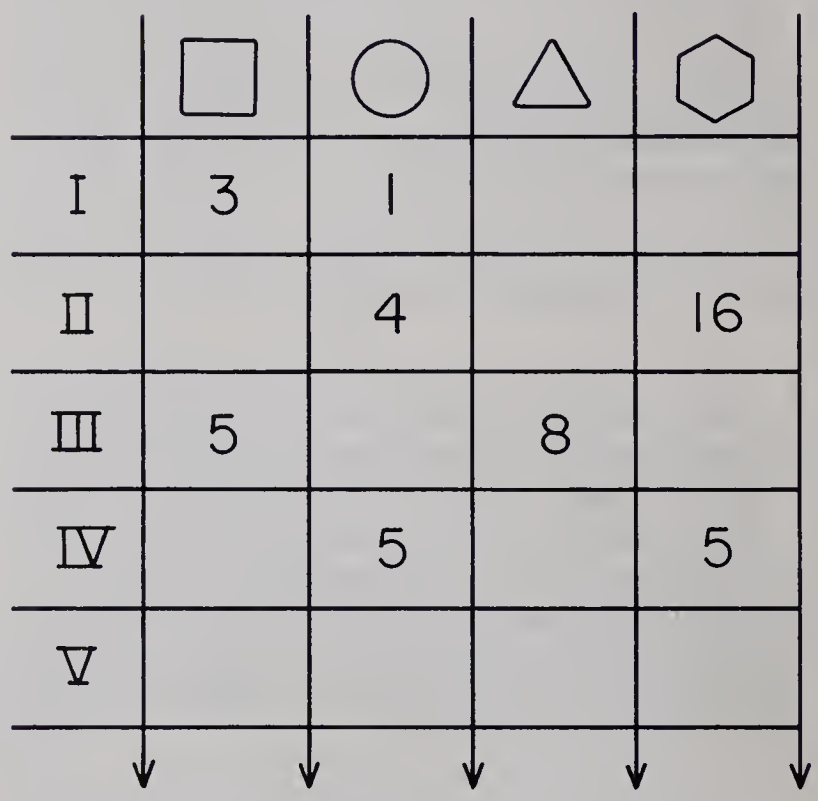
(When the product is a prime number, and it is given, then the pair must be composed of that number and one—the only factors of a prime number.)

In yet another form, the questions may look like this:



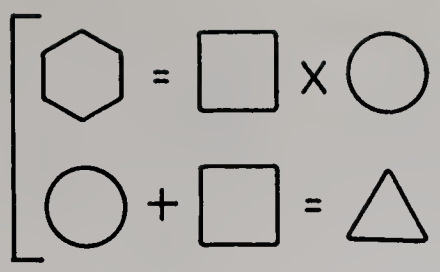
Now, an agreement is in order. In each example, any number that is indicated for a shape must occur in all similar shape's. The rules do not prevent the same number occurring in different shapes.



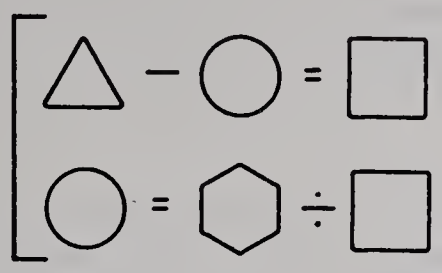


In example I, we are told that 3 and 1 are the pair. Then the sum, or 4, is recorded under the triangle; and the product, or 3, is recorded under the next hexagon. In example II, we are given one member of a pair (4) and the product (16). Since $4 \times 4 = 16$, the missing member of the pair must be 4; then the sum must be 8. In example III, the pair is obviously 5, 3 and the missing product is 15. In example IV, the product is 5 and one member of the pair, taken as a factor, is 5; therefore, the other member of the pair must be 1; and the sum must be 6.

An advantage of this presentation is that the questions may be asked in a variety of ways:



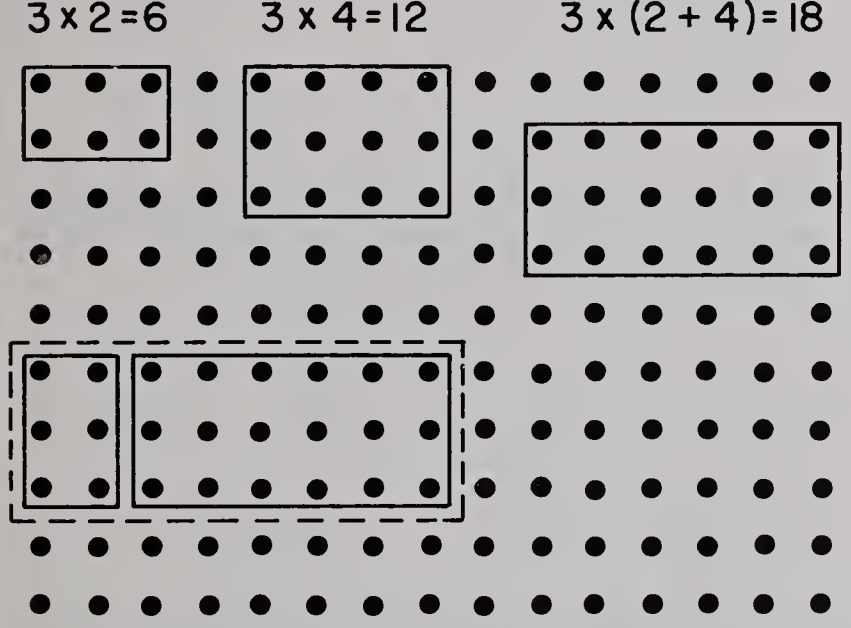
And, later, the same forms can be used in subtraction and division:



3. Pictures relating addition and multiplication

Begin with a large array of dots on the blackboard. Ask such questions as the following:

"Can someone draw a line around a group of dots to suggest 3×2?"
"Can someone draw a line around a group of dots to suggest 3×4?"
"Can someone draw a line around a group of dots to suggest 3 × (2 + 4)?"



After several similar examples, put the following three statements on the board:

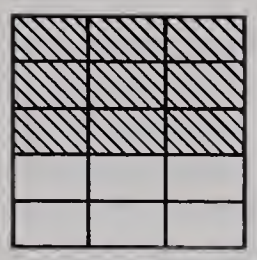
$2 \times 3 =$ ____
$6 \times 3 =$ ____
$(2 + 6) \times 3 =$ ____

"We can use one group of dots to suggest these three sentences. (On the board, mark off the section above that has been outlined with a broken line.) The section I have outlined on the board clearly suggests every part of the long mathematical sentence." Now, let's complete the sentences:

$2 \times 3 = 6$
$6 \times 3 = 18$
$(2+6) \times 3 = 2 \times 3 + 6 \times 3 = 6 + 18 = 24$

Next, draw several diagrams on the board and ask the children what mathematical sentences each suggests.

II



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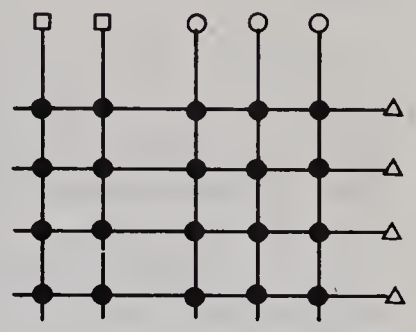

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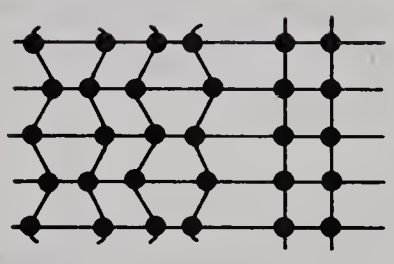
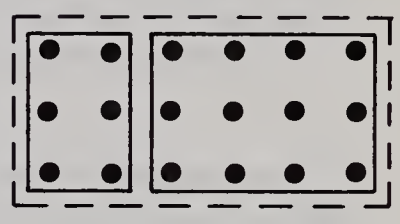


III



Mathematical sentences suggested by diagram I would include:

$4 \times 5 =$ ____	$12 + 8 =$ ____
$5 \times 4 =$ ____	$8 + 12 =$ ____
$4 \times 3 + 4 \times 2 =$ ____	$20 - 8 =$ ____
$20 \div 3 =$ ____	$20 - 12 =$ ____
$4 = 20 \div$ ____	$(12 + 8) \div 4 =$ ____

Similar sentences are suggested by each of the other diagrams. After sufficient sentences for each diagram have been noted, reverse the procedure by asking pupils to find the diagram that is suggested by a mathematical statement. "Which of the diagrams (shown above) is suggested by:

$3 \times 5 =$ ____	$5 =$ ____ $\div 6$
$18 \div 3 =$ ____	____ $- 6 = 12$
$2 \times 3 + 3 \times 3 =$ ____	$10 + 20 =$ ____
____ $- 3 \times 4 = 2 \times 3$	$5 \times 6 = 5 \times (2 +$ ____ $)$
$6 \times 3 = 6 +$ ____	$20 - 3 \times$ ____ $= 2 \times 4$

Have pupils volunteer to make up sentences suggested by the diagrams. The rest of the class must tell which diagram is being referred to in each sentence.

4. Games of logic

These games can be played many times with the children. With them, they learn to reason mathe-

matically, and then to extend their ability at logic to other areas. Each "If..., then...," statement employs mathematical logic or reasoning. Many of these games are also exercises in precise expression. The games are not fun unless we are sure that everyone says exactly what he means. This gives you, the teacher, an opportunity to emphasize the need, in certain circumstances, for precise expression of ideas.

a. ***What's My Rule?*** In this game of logic, the leader first asks for a number to which he then applies a rule, such as *add 2; subtract 3;* or *multiply by 10.* Others submit numbers to which the leader applies his rule until someone thinks he knows the rule. He must prove his discovery by giving the second number when another number is submitted. Examples:

Numbers submitted by others–	1	3	10	8
Response of leader–	2	4	11	9

(Rule: *The next number, or one more.)*

Numbers submitted by others–	2	7	5	12
Response of leader–	4	14	10	24

(Rule: *Twice the number.)*

Below are listed some of the rules which might be applied to the numbers 2 and 4 in order to get a third number:

2,4	2	(the *smaller,* or *first,* of the pair)
2,4	6	(their sum)
2,4	8	(their product)
2,4	3	(their average)
2,4	4	(the larger, or second, number)
2,4	5	(the second number and 1)
2,4	24	(the numbers side by side)
2,4	12	(twice their sum)

It is also possible to move out of arithmetic with this form of "What's My Rule?" Put the following on the board. Can the children finish the examples and make up some of their own that follow the same pattern?

1) a→b
b→c
m→n
s→__
__→s
__→r
__→__
__→__

2) a→c
b→d
f→h
r→__
__→x
__→f
__→__
__→__

3) a→A
b→B
c→C
m→__
__→X
__→Y
__→__
__→__

4) c→b
e→d
t→s
d→__
m→__
__→h
__→i
__→__
__→__

5) c→a
h→f
n→l
r→__
q→__
__→__
__→__
__→__

6) a→B
f→G
r→S
b→__
d→__
__→T
__→Q
__→__
__→__

7) band→and
bone→one
bin→in
bark→_____
_____→owl

8) 23→3
37→7
10→0
99→__
__→8

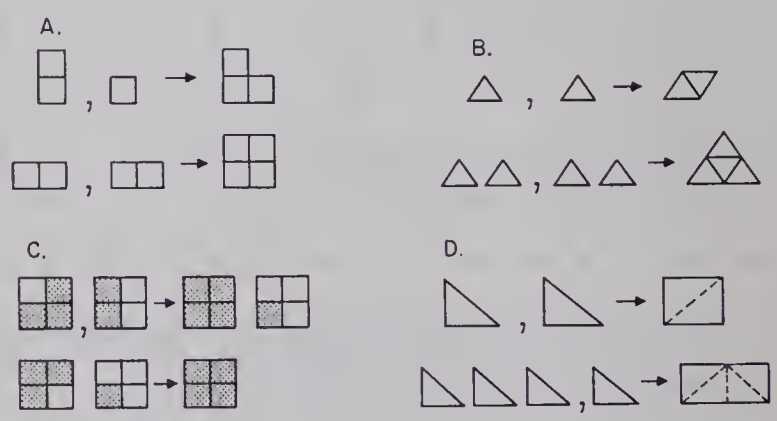
9) man→men
bird→birds
woman→women
mouse→_____
_____→feet

10) (present→past)
give→gave
have→had
hit→hit
run→_____
_____→fought
_____→_____

11) (reverse first and last letters)
now→won
saw→was
rat→tar
tip→____
pet→__
____→____

12) (reverse order)
123→321
37→ 73
8→ 8
25→__
____→34
____→____
____→____

Below are various examples that will help pupils see the relationship between the sizes and shapes of things:



E.
be, at→beat
my, self→myself
of, ten→often

F.
upon, on→up
into, in→to
today, to→day

G.
bother, am→wing
five, two→a
mother, up→year

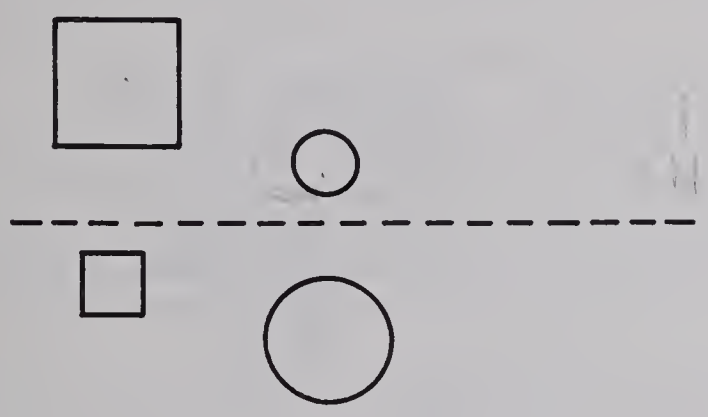
H.
mask, m→ask
tear, r→tea
swing, w→sing

In each of the above cases, a rule or a pattern is suggested by a few examples. As soon as pupils recognize the rule, they can complete other examples according to the rule, and they can make up examples of their own. There is no need to verbalize the rule—simply continue to illustrate it in examples.

The first child to understand the rule can be made an "expert" to test others who think they have discovered it. When two or three children have become experts, the teacher can help those who are still not tuned in, while the experts continue to test others.

Let children make up their own series to suggest a rule they have in mind. They can put their examples on the board, and act as "experts" for those who think they have discovered the secret. In this way, many activities can be going on at once without a teacher's direction, and the teacher can see that everyone succeeds in discovering some patterns.

b. *Fewest Questions.* Here is a game that strengthens number concepts, expands vocabulary and develops powers of reasoning. It is called "Fewest Questions." You can start with a very simple example on the blackboard or on a piece of oak tag.



Some preliminary questions might be: "How many shapes are there above the line? How many are there below the line? How many are there altogether? How many are squares? How many are circles? How many are large? How many are small?" Now to the game.

"I'm thinking of one of those shapes. Try to guess which one it is. We'll keep score of how many questions you ask."

Select a child as scorekeeper at the board to make a tally each time the guesser asks a question. Except at the outset, discourage having to point to the diagram in asking questions.

"Is it the big square?"

"No, it's not the big square. That's your first question."

"Is it the big circle?"

"No, it's not the big circle. That's your second question."

"Is it the little circle?"

"No, it's not the little circle. That's your third question. Do you know the shape I'm thinking of?"

Of course, if the guesser remembers the questions, he would know that it has to be the "little square" but he may keep on asking until he gets the answer.

After a few such guessing sequences (in which you may take the liberty of saying *No* as often as you logically can), offer to be the *guesser* yourself.

"I'm going to turn my back. Barney will come up to the board and point to one of the shapes. You will all know the one he selects—but I won't. Then I'm going to find out by asking just two questions!"

Your questions may be:
1) Is it a square?
2) Is it above the line?

There are only four possible combinations of answers. They are

	A	B	C	D
Is it a square?	Y	Y	N	N
Is it above the line?	Y	N	Y	N

If the answers to both your questions are "Yes" (A)—it is the large square.

If the answers are "Yes" and "No" (B)—it is the little square.

If the answers are "No" and "Yes" (C)—it is the small circle.

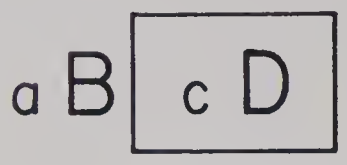
If the answers are both "No" (D)—it is the large circle.

You might ask another pair of questions.
1) Is is small?
2) Is it a circle?

Again, any combination of Yes-No answers will reveal the chosen shape. The secret is that each question must divide the group in half in a different way.

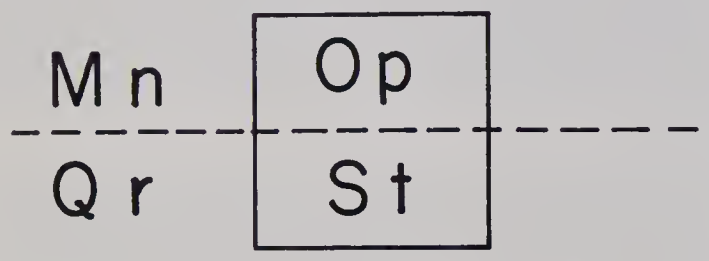
After this, you should return to the children

as *guessers* and *selectors*. (Some will get the idea right away. They've discovered the secret and they must keep it.) Change the diagram.



1) Is it a large letter?
2) Is it in the box?

After the children become adept at asking the questions so they never need more than two, enlarge the diagram.



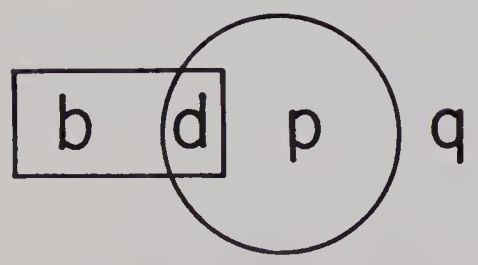
"How many large letters? How many small letters? How many letters altogether? How many letters above the line, how many letters below the line? How many inside the box? How many letters are both above the line and in the box? (Only two—O and P.) How many large letters are both below the line and not in the box? (Only Q.)"

That last question suggests a group of three questions that will reveal any choice:

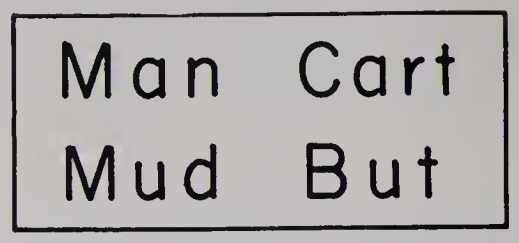
1) Is it a large letter?
2) Is it above the line?
3) Is it in the box?

(1)	Y	N	Y	N	Y	N	Y	N
(2)	Y	Y	N	N	Y	Y	N	N
(3)	Y	Y	Y	Y	N	N	N	N
	O	p	S	t	M	n	Q	r

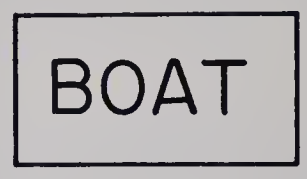
"Fewest Questions" is a favorite with the children. Let them take their time in discovering the secret. You may vary the objects to include geometric shapes, letters of the alphabet, short words, numberals, etc. Below are a few variations in design with examples of the fewest questions that will expose any item chosen.



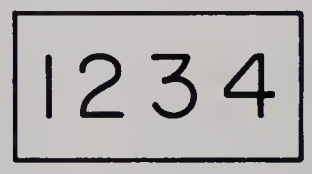
1) Is it in the rectangle?
2) Is it in the circle?



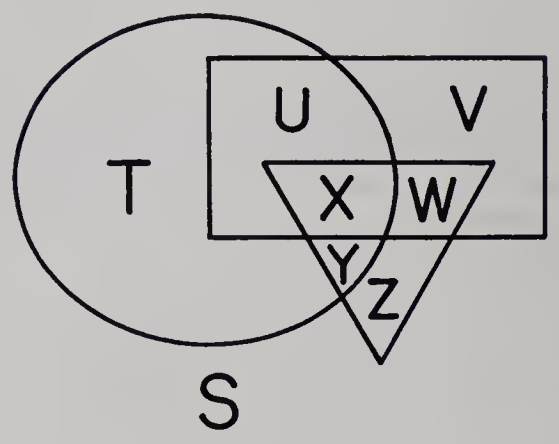
1) Does it begin with *M?*
2) Is there a *u* in the word?



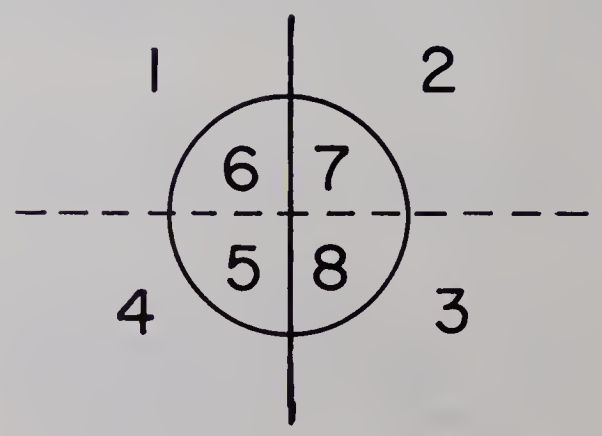
1) Is it in the word BARK?
2) Is it in the word TUB?



1) Is it one or four?
2) Is it more than two?



1) Is it in the circle?
2) Is it in the square?
3) Is it in the triangle?



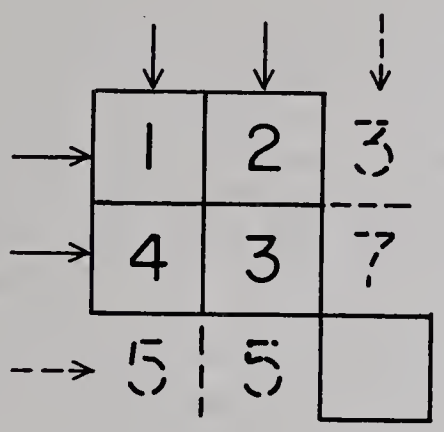
1) Is it above the line of dashes.
2) Is it to the right of the solid straight line?
3) Is it in the circle?

E T A O I N S H

1) Is it in the word WINTER?
2) Is it in the word FINISH?
3) Is it in the word EASIER?

5. The Cross-Number Puzzle

On the board, draw the following:



The arrows, you will explain, tell us to *add.* As results of the four examples of addition indicated by the solid arrows are recorded, we are led to wonder where to record the additions indicated by the broken-line arrows.

Of course, in one example shown, we could write 10 in the box and it would serve jointly as the result of 3 + 7 and of 5 + 5. Is this a coincidence? Most children and adults think it is, and that this outcome is dependent upon the choice of the four numbers originally written in the box.

Many of your children would be less surprised by the outcome if they imagined objects in the four cells—one dot, two squares, three circles, and four cars. We can sweep the objects across into piles of three and seven; then sweep them down into a single pile of ten. Or, we could sweep them down into piles of five and five; then sweep them across into a single pile of ten.

If we started with ten objects and swept them together we would not expect any to appear out of thin air or to miraculously disappear. If we start with ten objects, we will end up with ten objects.

We might keep track of the goings-on this way:

$$(1+2)+(4+3)=3+7=10$$

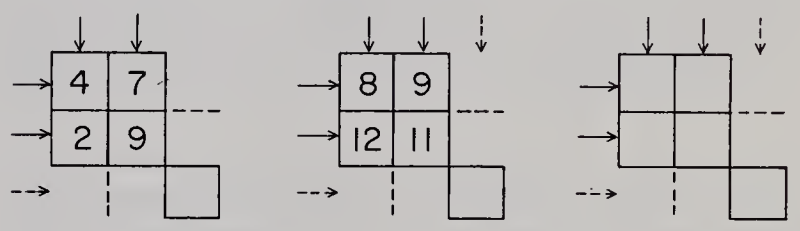
If we had, however, swept down both columns and then across the bottom row, we might keep the following record.

$$(1+4)+(2+3)=5+5=10$$

There is a fundamental rule in arithmetic that the order in which numbers are added does not affect the result. In columns of addition, we often "add up" the columns and then, to check our computations, we "add down" the columns, because we are sure that the order in which we add ought not to change the result.

Yet, when most children and most adults are first confronted with a Cross-Number Puzzle, they are amazed that the box in the lower right-hand corner will suffice for recording the addition in both the right-hand column and the bottom row.

Try the next two examples and then write any four numbers that come to mind in the third:

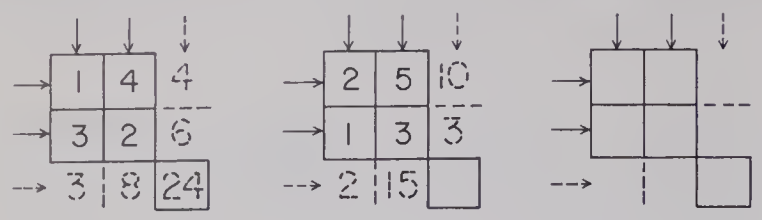


You probably have several children in your class who can give this puzzle a real work-out while trying to discover a group of four numbers that won't work. Even though they have swept objects on the floor and visually swept designs on paper, first one way and then the other, they do not fully comprehend that those numbers in the cells are answers to "How many?" How many what? Anything you like or don't like, or that can be counted.

a. In-any-order rule. In this work, you are helping children see examples of a basic idea in mathematics: *When we add, the order in which we proceed is immaterial.* We have nicknamed this rule the *in-any-order* rule. (Mathematicians will analyze it more carefully and tell us we are combining two fundamental laws, the Associative law for addition and the Commutative law for addition. The *in-any-order* rule for addition seems more descriptive and is easier to remember.)

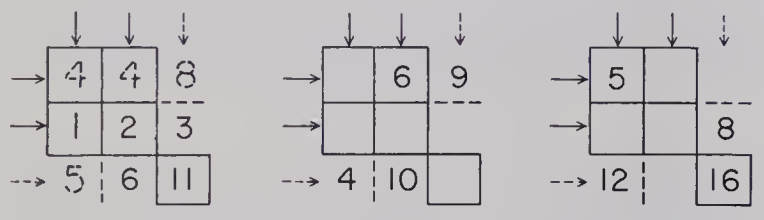
b. In-any-order works in multiplication. This same puzzle helps reveal that there is an *in-any-order* rule for multiplication, too. In the examples below, the arrows tell us to *multiply.* (In the third example, try any four numbers that come to mind.) Children will be helped by this puzzle form to learn what mathematicians call the Commutative law for multiplication and the Associated law for multiplication which we, for the sake of clarity, brevity

and usefulness, combine into the *in-any-order* law for multiplication.



c. Subtraction—the inverse of addition. The Cross-Number Puzzle will also be used to present subtraction as the twin of addition, its opposite or *inverse.* The agreement for the puzzle below is, as earlier, that the arrows indicate addition and that the bottom right total is the result of adding the four numbers in the basic four cells. We are now faced with a problem of reconstruction, or building backwards.

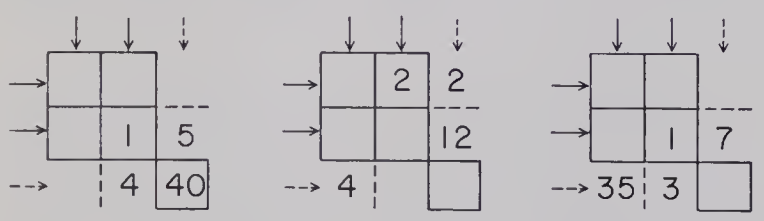
Addition:



To rebuild, in the first example, you may argue: I'm told that ____ + 3 = 11 or in other words 11 − 3 = ____. I know that 8 + 3 = 11 − 3 = 8. Therefore, "8" must go in the top cell on the right column of the first example.

d. Division—the inverse of multiplication. Similarly, we can reconstruct puzzles when the rule is that the arrows indicate multiplication.

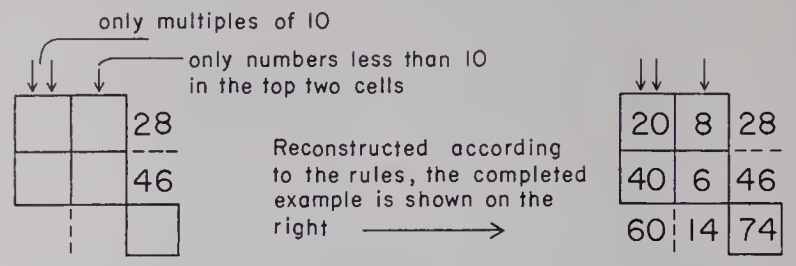
Multiplication:



In the first example the question is asked, "What number multiplied by 5 results in 40?" Or, "40 divided by 5 equals what number?" Etc.

The problem of reconstruction now involves *division.* Again, we have a dramatic example of division as the opposite of multiplication.

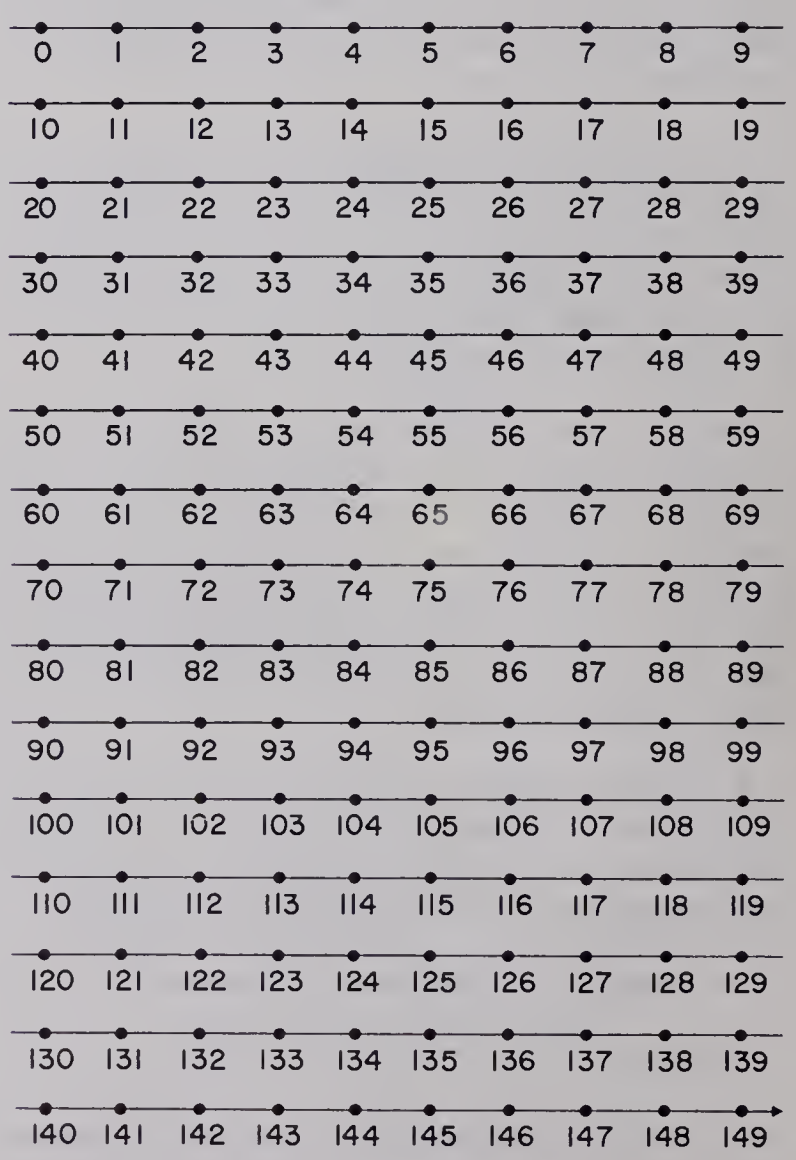
e. Help with algorisims. Cross-Number puzzles help with the algorisms children will later learn to employ when necessary. The examples below call for reconstruction according to the agreement that, in the column under the double arrow, only multiples of 10 can be used and in the column under the single arrow, only numbers less than 10 can be used in the top two cells. Otherwise, the normal rules of addition are used.


only multiples of 10
only numbers less than 10 in the top two cells
28
46
Reconstructed according to the rules, the completed example is shown on the right
20 8 28
40 6 46
60 14 74

In following the rules, 28 and 46 are "regrouped" and then added in this regrouped form. This series of steps will be condensed or abbreviated and grow into the algorism of addition which we know.

6. Number-Line games

A large copy of a number-line chart like that shown below can be put to good use in the classroom. It can be used to demonstrate that it is possible to jump from 2 to 12, or 5 to 15, or 7 to 37 with great ease. Pupils won't need to count in order to do this, either. Every jump down "↓" is equivalent to adding "10." A jump down "↓" and ↷2↷ is equivalent to adding "12." A jump down "↓" and ↶2↷ is equivalent to adding "8." To find the sum of 10 and 17, you start at 10, jump ↓ one row and ↷7↷ and, magically, you're on 17. Or, start at 10, jump ↷7↷, and then jump ↓ one row.



Allow your pupils to spend several days in learning how to move about the number-line grid. They will enjoy exercises like these: "Where will you land if you start at 0 and take three jumps down? Four jumps down? Two jumps down? Six jumps down? Ten jumps down?

"How many ways can we make the trip from 11 to 22? From 5 to 12? From 3 to 24? From 7 to 18? From 25 to 37? From 111 to 123?

"Where are we if we start on 15, jump down two rows and four jumps to the right? (15 + 20 + 4 = 39) If we start at 15, take four jumps to the right, and jump down two rows? (15 + 4 + 20 = 39) If we take two jumps to the right, jump down two rows, and jump two to the right? (15 + 2 + 20 + 2 = 39)"

Have pupils jump forward or back, and then up, to subtract on the number line. "Just as there are several ways to get from 15 to 39, there are also several ways to get from 39 to 15."

When pupils understand how to use the number-line chart, it is possible to play games with it. For instance, one game is started by saying, "I'm thinking of a number. Start at 15, jump three rows down and four to the left. What is the number?" Whoever can tell you first what the number is becomes the leader and has to think of a number for which he must give appropriate clues. Some pupils may become so proficient that they won't need to use the number-line chart. Encourage such pupils to "do it in their heads."

Other games could proceed like this: "Start at 0, add 11. Add 11 more. Add 11 more. Where are you?" Or, "Start at 3, add 6. Add 3 more. Add 6 more. Where are you?" Use repeated additions and subtractions of 3,4,5,6,7, etc, and ask pupils to look for patterns as they progress.

Jumping on the number line can be used to help underline the fact that in subtraction we simply *undo* what we *do* in addition. If *jump down one row* is equivalent to adding 10, then *jump up one row* must be equivalent to *subtracting 10.* If a jump of *two to the right* is equivalent to *adding* 2, then a jump of *two to the left* is equivalent to *subtracting* 2.

a. Regrouping numbers. In order for a pupil to add 8, it is possible to add 10 and subtract 2, or to subtract 2 and add 10. Confronted with 17 + 8, pupils can find the sum by regrouping the 8 in such a way that the sentence becomes: 17 + 10 − 2, or 17 − 2 + 10.

It is not possible to outline a set of rules for regrouping numbers. As each pupil gains confidence in putting the idea to use, he will also gain an appreciation of when regrouping is especially helpful to him in simplifying a problem. As they jump back and forth on the number line chunks to add and subtract, children learn to feel what's going on, because they *see* what they are doing.

To subtract 3 from 61 on the continuous number line is simply a matter of taking a jump of 3 backwards, but on the number-line chart 61−3 must be handled differently. The problem has to be approached as 61−10 + 7, or 61 + 7 − 10.

b. The decimal system and shortcuts. The structure of the decimal system is emphasized in the way the number-line chart form is designed. Bases for understanding the algorisms of adding and subtracting can be firmly established by developing the ability to jump freely on the chart. Enterprising children will discover many shortcuts for making jumps. The search for shortcuts should be encouraged, because in discovering shortcuts, pupils also discover basic patterns in the decimal system.

7. Discovering matching properties

Research on *discovery in arithmetic* has revealed important information. When children discover a relationship, they are excited and anxious to stay with that discovery. They will eagerly put this new idea to work, showing more examples of *it.* They come alive, often with an "Oo-o-o! I've got it!" or with a burst of industry using a newly discovered relationship to reel off answers with confidence.

What is the *it* they have glimpsed? Well, if we distract their attention with a request for them to *tell about it* or *put it in words,* we have selected a method that could not be better calculated to mutilate the experience.

That a child can move ahead, putting *it* to work successfully is certainly all the evidence we need to know he *has it.* Our only justification for veering off in the direction of a semantic exercise is a feeling that *it* is a generalization that ought to be verbalized. But research does not support this as a valid reason.

Transfer to future experiences is not increased by *verbalizing a generalization.* Sub-vocal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, the discovery of a new and important relationship, will stay around as an effective tool even though it hasn't been named and talked about. An opportunity to put this tool to work will contribute most to its staying power.

8. Silent teaching and discovery

The silent technique of instruction has revealed the mechanics of discovery unencumbered by any attempt at verbalization. A class can be kept on the edge of their seats for an uncommonly long time (and, incidentally, almost alarmingly quiet) as you move on from one level of difficulty and involvment to another. Though the way leads from the familiar deep into new, unfamiliar territory, interest is high, reactions quick, and with the teacher's eyes moving to the fingers, ordinarily timid children can be drawn into participation. The most retiring or placid child may, all at once, seem to come alive. Children who usually are ready first with the answers, are not as effective in keeping in the center of the spotlight.

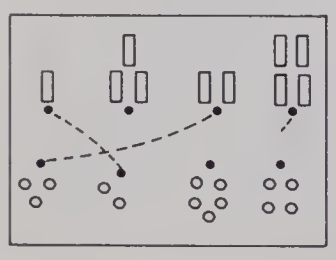
At the conclusion of a silent session, there is really nothing to talk about. And no one seems interested in talking about the experience. But as soon as you lose your voice again, there is an immediate awakening of the feeling of excitement. If a similar activity is initiated several days later, it will move faster than before. No generalizations were verbalized, yet something of what happened has been transferred from the trips into silence.

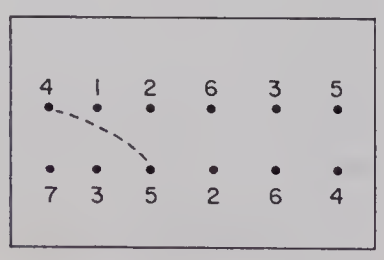
The only danger involved with arithmetic without words is that of over-stimulation. *Discovery* is not a relaxing state, physically or mentally. An act or moment of discovery is a moment of elation that seems to release a dynamics that can be almost self-consuming. The look of eagerness on so many faces all at once is infectious and it is tempting to see how fast and how far you can go.

The best way to assure a moderate forward movement is to concentrate on participation by everyone. If it's a board activity, everyone should handle the chalk. If it's a manipulation of blocks on a table, everyone can show he knows the secret.

One rule of teaching that will multiply the opportunities for discovery in any activity, arithmetic included, is to use the fewest possible words. And when you must talk, leading questions are much more fraught with educative value than any explanations you can make.

a. The game of "Next After." Without any words, put the following examples on the board:





The rule you have in mind is, of course: *One more,* or *the next label on the number line in a direction away from zero.*

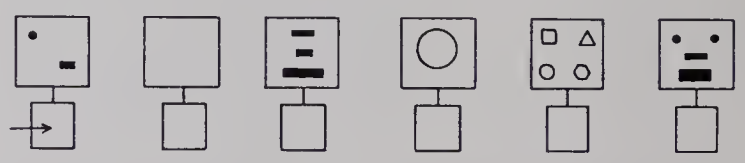
Complete the first connecting line, one square to two circles. Begin the next line and stop, looking for volunteers. Many children may think you have made a mistake. You must show confidence in your connecting. Some volunteers may make unsuccessful guesses as to your secret rule. If no one guesses it, you may have to draw the connection between the dots associated with two squares and three circles.

Start a third connecting line from the dot under the four squares. At worst, a child will connect it to four circles. You are troubled. You retrace the first and second connecting lines. With the chalk, you touch the single square and then the two circles. You express approval. Then you touch the two squares and the three circles. Again you are pleased. And again you look for volunteers. You may even have to complete the entire first example by yourself, though this would be unusual.

Move on to the second example. If no one has read your mind in the first example, begin by connecting the dot associated with four in the top group and five in the bottom group. Proceed as in the first example. You will be surprised by the reaction of the children. This time they will get the idea.

Why? Experiments lead to the conclusion that children understand the "Next After" *numerical game* quickly when it is associated with a memorized sequence of words or symbols rather than in the context of a group that has one more member than another. If you encounter this development, place increased emphasis in counting collections of things and arranging them in order from smallest to largest and from largest to smallest.

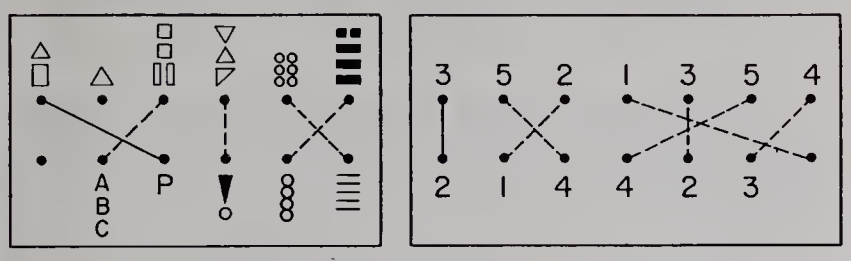
Erase the two examples and go back to labeling pictures of groups.



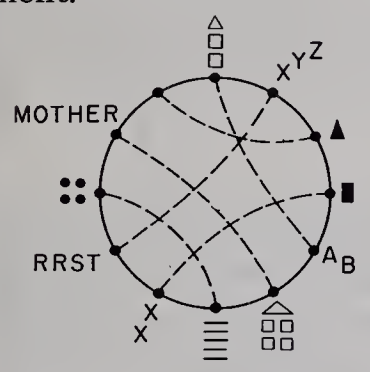
Then go on to another series of pictures. Skip the first, put one design in the second, two in the third, and stop just as you are about to start the fourth. Any volunteers, you seem to ask?

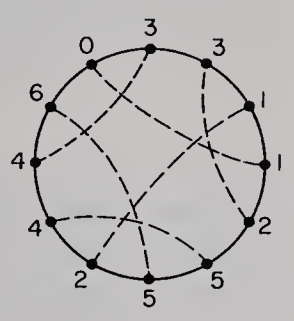
The idea is, of course, the same as with the original pair of examples: *one more,* in another context.

b. More silent teaching. If you feel that your class's response is advanced enough, go on to more advanced silent treatment examples. Use pairs of examples, one with designs, the other numerals.



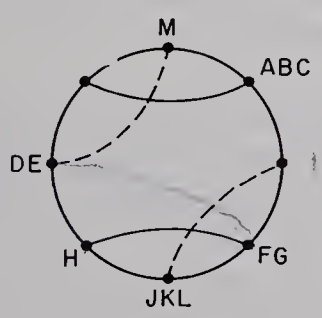
Starting at a dot associated with the pictures in the top row, draw a line to a dot above a group with one less in the bottom row. (Note that *0* or *none* is *one less than one.*) Then move to a circular arrangement:



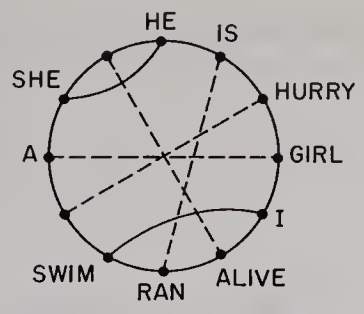


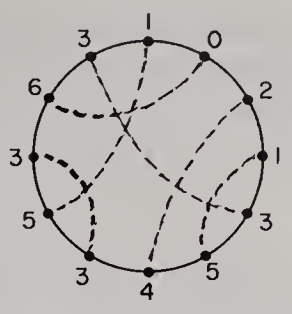
Now, one more and one less are two different ways of looking at a line. There is no general pattern of *from top to bottom* as in the first examples. Further, there are alternate ways of completing the connections. Both illustrations are quite similar. The second can be seen as recording the answers to *how many* designs at the corresponding point of the first. But, the connections are drawn to illustrate how different they may be in the two cases.

Jump to another rule. It also has a name, but there need be no hurry to label it.



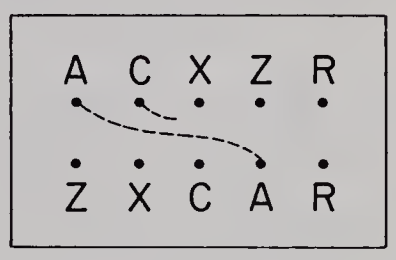
Make two connections, then look for volunteers. This is far from obvious, but it is one of the most important ideas in arithmetic. Our hope is that children will be able to read your mind and make right connections many times and in many contexts before we name the idea *addition.* Try an extension of the idea:

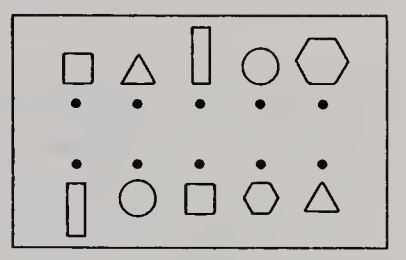




The secret in the first example is *five*; in the second, *six.* Some children may be thinking along these lines, but it would spoil the experience to talk about it. They are reading your mind. They glimpse a pattern that must parallel yours because they can help you draw the connections your patterns would dictate.

c. Matching designs. You have laryngitis again. Put examples such as the following on the board:

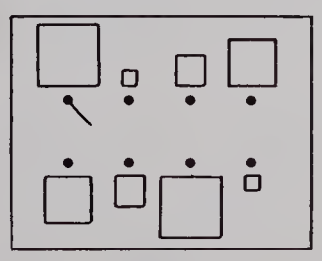


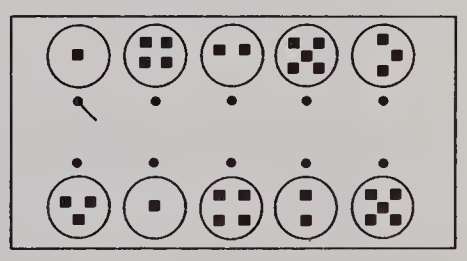


Start drawing a line from the dot under the "A" and stop. You don't have to do this job you were about to do. Look around for volunteers. There are probably several already. Hold out the chalk to one. If he continues the line you started with until it connects to the dot over "A" in the bottom row, you beam from ear to ear because your mental telepathy was successful.

If the volunteer does anything else, you are upset because your telepathic effort wasn't well enough carried out. Remember, you have not said a word, no one knows the rules for sure, no one can make a mistake. You have to try again, and again until you have *communicated.*

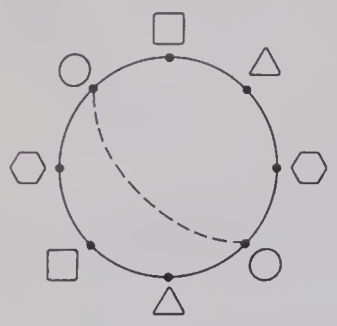
Start a line from the dot under the "C" in the top row. Again decide against extending that effort and look for volunteers. There will probably be many by now and you can move rapidly through both examples of matching. Next, try examples such as these:

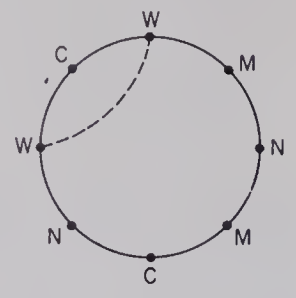




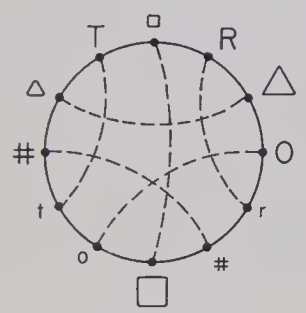
A similar activity, a bit more sophisticated,

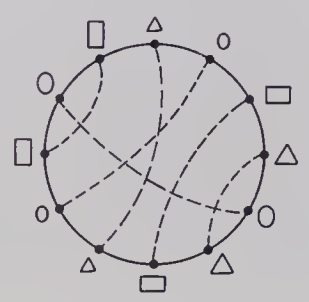
develops when the objects or pictures to be matched are not displayed in two rows but, rather, around a circle.



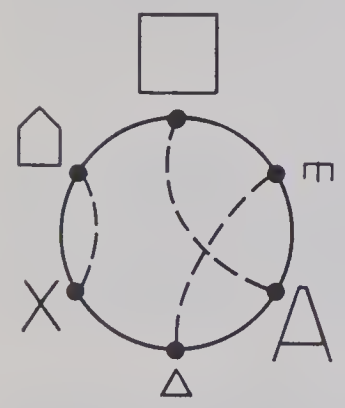


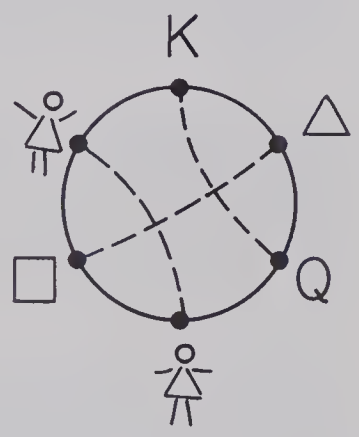
Then more matchings and more sophistication. (In the second example below, there are four of each kind of symbol, so the connections can be made in several ways, any one of which is as good as any other.)



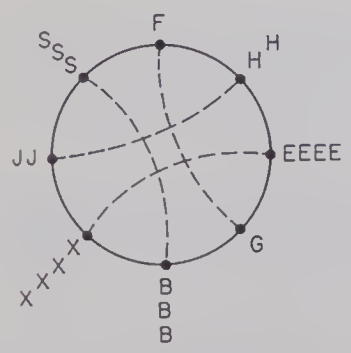


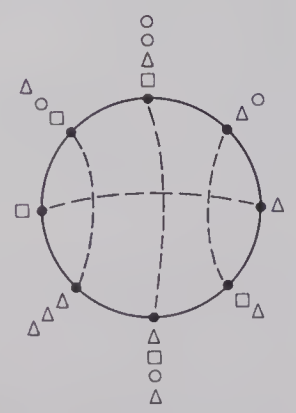
d. Largest, smallest, in-between. You still have acute laryngitis. In the first example below, largest, smallest and in-between, are the basis of matching. In the second, letters, shapes and people.



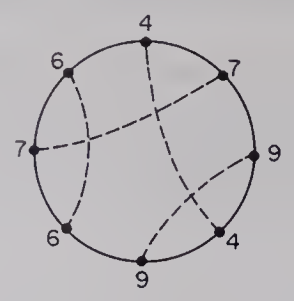


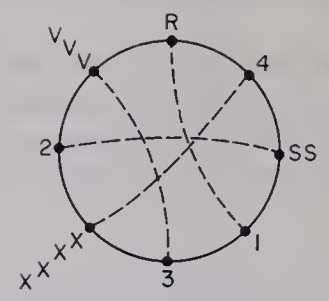
More examples:



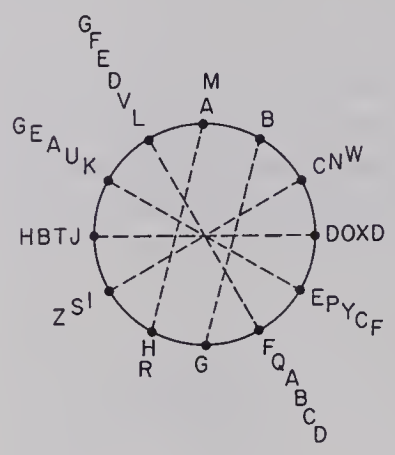


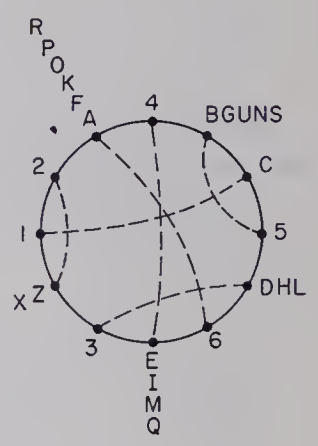
And on to the following:





You still haven't said a word, but activity is going ahead rapidly.





We have come a long way in a short time, moving through several very different kinds of matching, changing rules as we went, all without a word. This technique was first developed as a suggested answer to the question: "How could you teach arithmetic to young children if you did not speak nor understand their language?" Experiments led to the conclusion that you might find a real advantage in not being able to let the confusion of words distort the pictures.

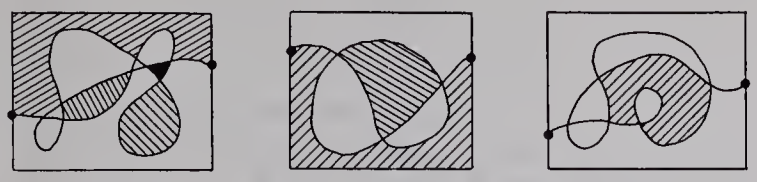
Number is an abstract idea. It is a *property* that groups of objects or other collections may have in common. If they do have this property in common, we say they "have the same number" and we assign a name and a symbol that serves to point to that particular *number* or *property that is common to a whole collection of groups.*

The sum total of the preceding paragraph illustrates what happens when we try to talk about the idea we were freely communicating to children before we resorted to *words.* Every matching was based on recognizing *common properties.* We ended with a matching between collections of letters and the symbol indicating their *number,* indicating the collection of collections to which they belong when number is the property we are noting.

9. The study of regions

Many experiences usually classified as art or "free expression" can be given a mathematical

orientation. For example, draw several rectangles on the board. Ask children to draw a continuous line from one side of the rectangle to the other, letting the line wander about the space. You may have results such as this:



Now, we must define a space or a region. Suppose the frame of the first rectangle is a fence and the lines we drew are fences, too. Imagine a cow eating grass. The cow is located at *a*. Color the area where there is grass the cow can eat.

In the next rectangle, imagine there are two cows. They are in different pastures. Where might they be? One might be at *b*. Color it in. (This area is chosen to demonstrate that the outside of the rectangle is also considered as a fence.) Where might the other cow be? Have children point to the area, and color it in.

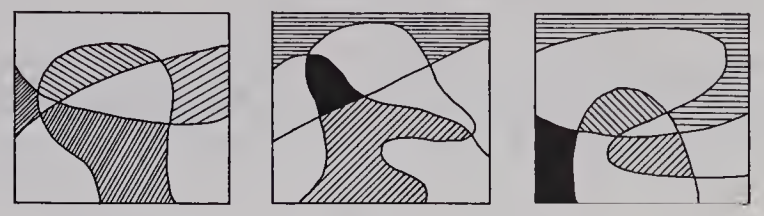
In the third rectangle, imagine there are four cows, each in a different pasture. Here are four different colored pieces of chalk. Use a different color for each pasture. (Four different children can work on this example, one after the other.)

Next, to individual work. Have everyone draw a wandering fence with a black crayon on a piece of paper. Have them draw another wandering fence in black crayon on the reverse side of their papers. On the first side of the paper, they are to imagine there are three cows, each in a different pasture. On the reverse side, they are to imagine there are four cows, each in a different pasture. They are then to color in the pastures where they imagine the cows to be.

The number of cows can, of course, be varied at will. After the activity has been tried once or twice, your directions may change. Two cows are in red pastures and three cows are in blue pastures. With a red and a blue crayon, the children should color in those pastures. Write a summary of instructions on the board:

2 red pastures
3 blue pastures

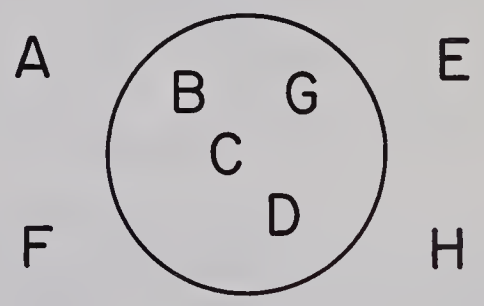
Here is another way to get a design started. Draw several rectangles on the board again. Have one child after another follow these directions. (1) Draw a line that starts at one side and goes across to the other but doesn't cross itself. (2) Draw another line that goes from one side to the other and crosses the first line. (3) Draw a loop that starts at the bottom, goes up and then comes back to the bottom (or from top back to the top). Examples might look like this:



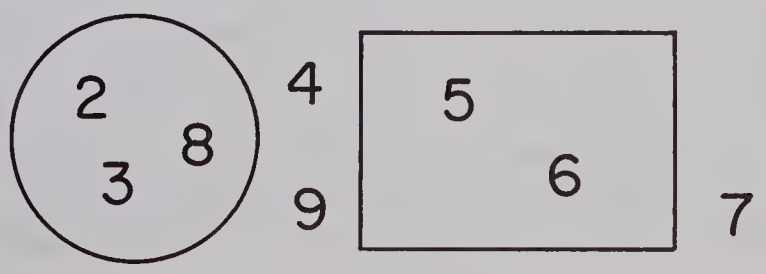
With different colored chalk, shade in any desired number of areas, regions or pastures. Now, as before, all the children can work independently. On one side of their papers, they might be asked to color three pastures or regions one color and three another color. On the other side, they might be asked to color six regions different colors, or all in the same color.

You may want to have the children write a number in the lower right-hand corner of their papers that tells how many spaces are colored—a title for the design. Drawing straight lines, curved lines, lines that cross, seeing intersecting lines create interesting regions are all ideas that will extend into the study of spaces that we call geometry. Following simple directions that lead to surprisingly different results each time, is a valuable experience.

a. Outside and inside. Explain to the class that a single closed curve of any kind establishes two *regions:* the *outside* and the *inside.* To demonstrate this, draw the following diagram on the board:

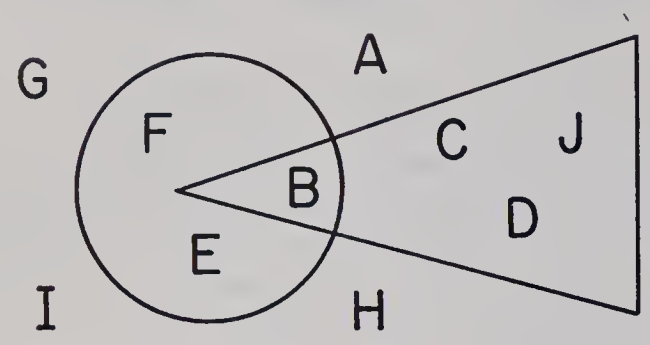


Ask, "Which letters are in the *inside region*?" Which letters are in the *outside region*?" Next, explain that the number of regions to be considered increases if more than one figure is involved. Draw the following diagram of a circle and a square on the board:



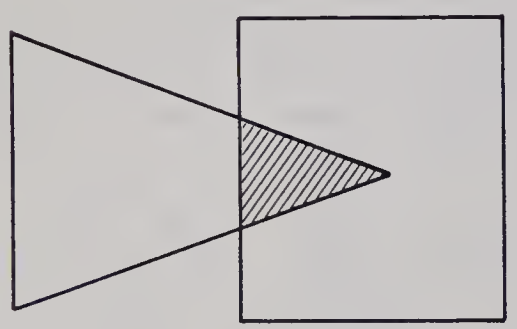
Ask, "How many numerals are in the region outside the circle? (5 and 6 are *inside the square, but they are also outside the circle.* The answer, therefore, is 6.) How many numerals are inside the square? (2) How many numerals are either inside the square or inside the circle? (5)"

Overlapping shapes make it especially necessary to be precise about what region is being discussed. Draw the diagram:



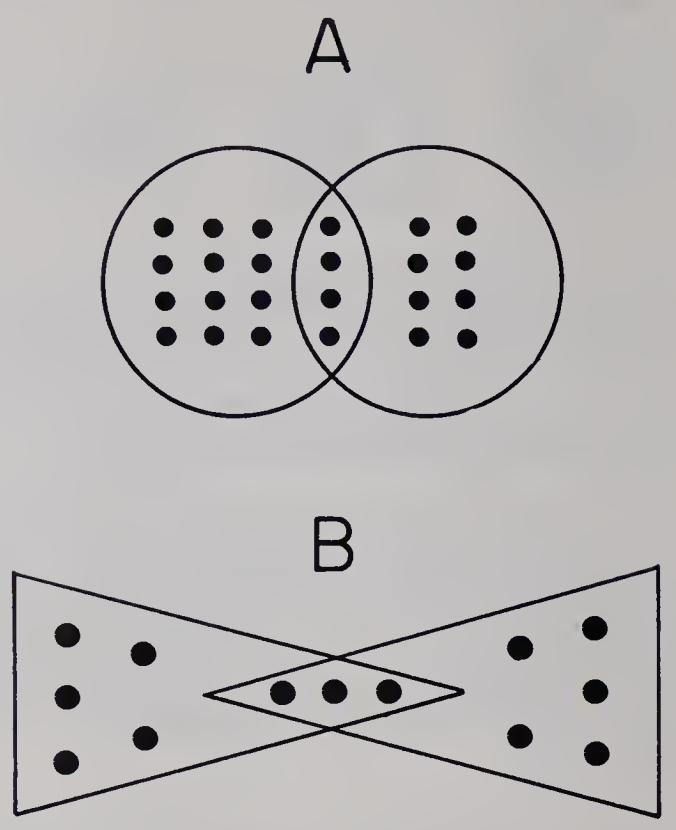
Ask, "How many letters are inside the circle? (3) How many letters are outside the circle? (7) How many letters are inside *both* the circle and the triangle? (1, "B") How many letters are *not* inside *both* the circle and the triangle? (9—all, except "B") How many letters are inside *either* the circle or the triangle? (6) How many letters are *outside* both the triangle and circle? (4) How many letters are *inside* the triangle but not inside the circle? (3)."

Draw the following diagram on the board exactly as it is shown here and have each pupil copy it:



Pupils should use a red crayon to color the region that is *both* inside the triangle and inside the rectangle. Then have them color the region that is *inside the rectangle but not inside the triangle* with a blue crayon. They should then color the region that is *outside* the rectangle with a green crayon. This latter region would include all of the triangle that does not overlap with the rectangle.

b. "I'm thinking of a region." Using a different colored chalk for each *region,* draw the diagram that is shown below on the board. Then, play the game "I'm thinking of a region," using this diagram as a model. Any region can be identified by the shape or shapes that form it and by the colored chalk used to delineate it. The object of the game is to guess the region the *leader* has in mind and to do so by asking a minimum of questions. Some pupils will discover the technique of cutting down the possibilities quickly, and will master the game easily.



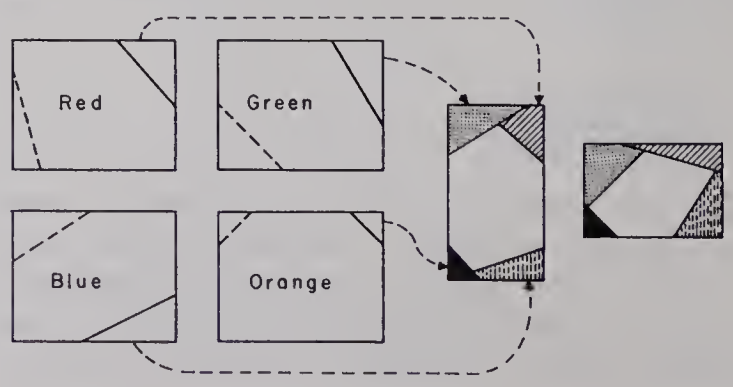
10. Making triangle designs

Suppose now for a moment that we had four pieces of colored paper—red, blue, green and orange. We could cut one corner off each sheet, some big pieces and some smaller pieces, and paste these four corner pieces on a new sheet, fitting the corners we cut off into the corners of the new sheet.

"Here is a picture of four sheets of colored paper. Who can draw a line on the sheet marked red to indicate a corner that is cut off?" Ask the same question for each sheet. Urge some children to make big pieces, small pieces, long thin pieces, etc.

"Now, let's look at the first picture frame. It's empty. What would it look like if we pasted that red corner on the picture, fitting it neatly in the corner?" Do the same with other four colors.

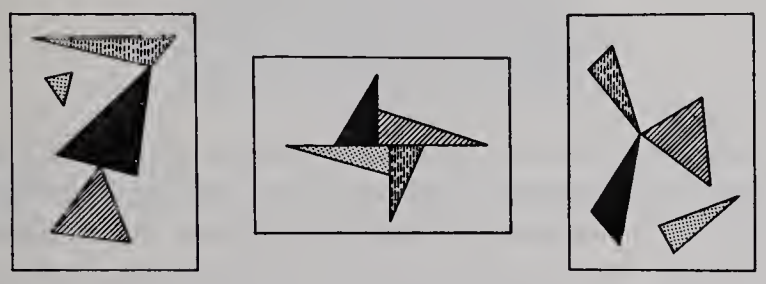
Repeat the activity, starting with a line that indicates another corner cut off the red piece. Construct a second design.



Let the children repeat the activity working either individually or in small groups.

Right triangles and similar triangles. The corners with which the children are working will later assume the dignified name of *right triangles.* The idea of *fitting* as it is used here becomes very important when we want to talk about *angles of the same size,* a condition which exists when one angle can be fitted into the other. When each of the angles of a triangle can be *fitted* into an angle of another triangle, we call them *similar.* The idea of angles fitting or not fitting is a required concept when talking about a shape or comparing shapes.

When the children have finished their corner-frames, there is a space left in each in which they can draw any pictures they like. Either at this time or later, other triangles, shapes with three sides, shapes with three corners or three points, can be cut from the remnants of the colored paper to make another kind of design. Put several examples of *triangle* designs on the board.

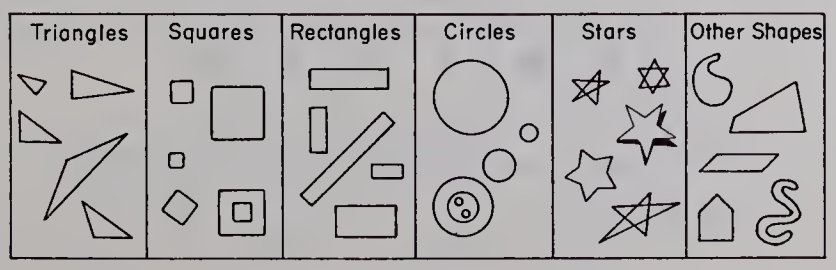


11. Drawing shapes

Section the board and write a name in each—triangles, circles, squares, rectangles, stars, other shapes.

"This section is for triangles. Who wants to draw a triangle in this section? Another triangle? A long thin triangle? A small triangle? A triangle with a very sharp point? A triangle with two very sharp, thin points?"

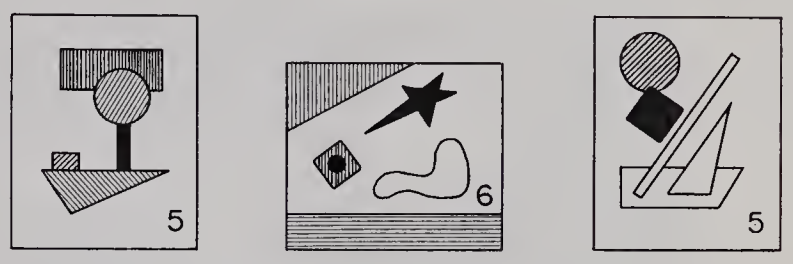
Then to *squares.* Bigger squares, smaller squares, a square inside of a square, etc. Whenever a volunteer draws a shape that doesn't belong, talk about it. "Does everyone think this shape is in the right place? Where should it go?" If it can't go in any other specific section, it can always go in the section titled "Other Shapes." No contribution need be excluded; some merely have to be redrawn in another section.



Now ask each child to draw at least one shape for every section—a triangle, square, etc. Then have them cut them out. Encourage the children to use different colored paper and to see if they can use up all of their scrap pieces.

Making designs. "Pick five of the shapes. Paste them on a sheet to make a design you like. Then take six more shapes and make another design. In the right-hand corner at the bottom, write the number that tells how many shapes there are in the picture."

On the board, sketch examples; or have some of the children sketch them. (Use several different colors of chalk.)



Have each child select a design he has made, the one he likes best. Line them up on the chalk rail or on a bulletin board. Give each a letter or number designation.

Pupils choose best design. The children are a jury. The job of each child is to study all the designs and pick the three he likes best. He is asked to write the letters or numbers of these three on a piece of paper and put it in a ballot box.

A committee is chosen to record the votes, giving one point to each picture named on each ballot. The tallying is best handled by a small group and the several designs with the highest votes are displayed prominently. (In this way, no one need be embarrassed by a public announcement that his design had none or very few votes.)

Should you discuss the three or four or five selected designs? Very briefly and not in detail, if at all. Children are good art critics but they are not able to express their basis for preferring one design to another. They respond to balance, symmetry, movement, color harmonies, etc., without having the background to verbalize such reactions.

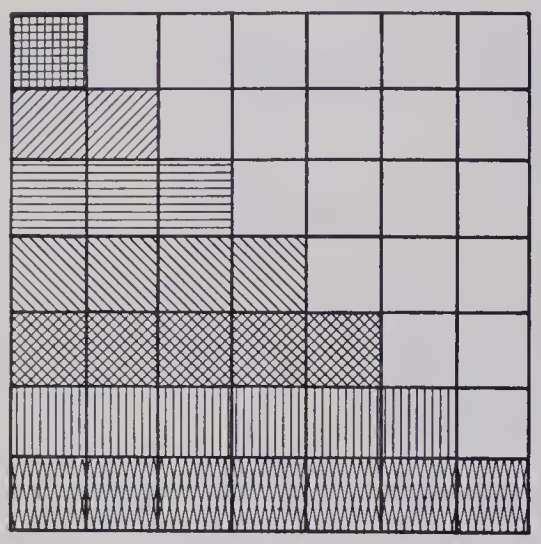
It is enough for the children to have participated in selecting favorites among a whole collection of abstract designs. There are no people or trees or cars that may or may not look like people or trees or cars to everyone. The children have made a judgment. *Likeness* was not the basis. What was the basis? You will never know, but it will be a mixture of balance, symmetry, movement, color

combinations and all the qualities that an art critic can verbalize.

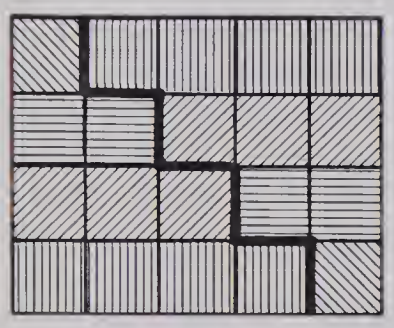
12. Graph-paper arithmetic patterns

Squared paper invites activity. It is never too early to begin; few persons will ever grow beyond graph-paper arithmetic—in mathematics, in engineering, in business, in reading the newspaper. No more effective methods have been discovered than work with graph paper to help reveal the simplest to the subtlest relationships.

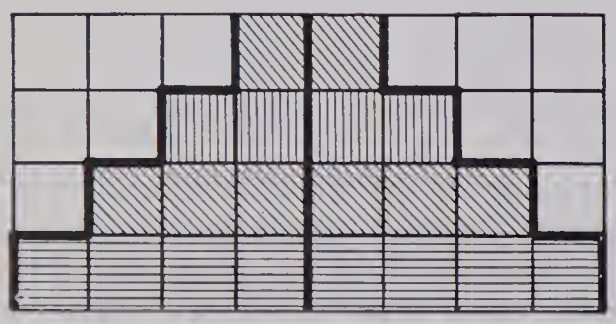
Stair-step patterns. Begin with the lowly *stair step.* Color a square with any crayon. Then the one under it and another one a different color, etc.



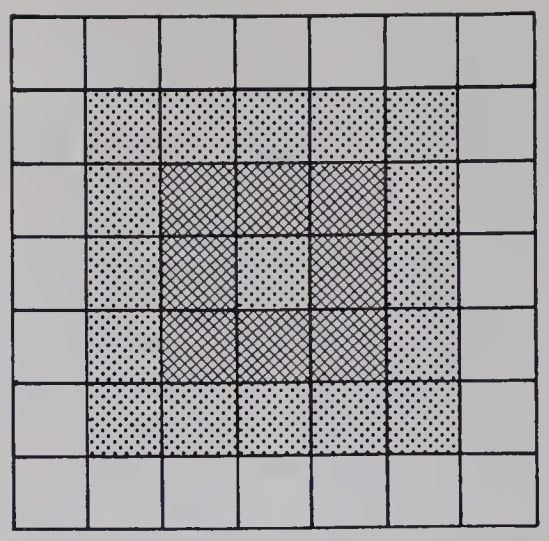
Make another set of *stair steps* up to ten high. Then, repeat the color combinations in an *upside down* set that fits on top.



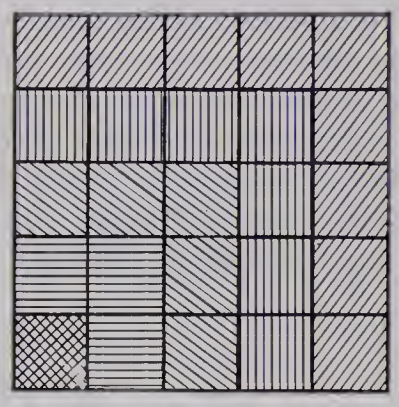
And, *back-to-back stairs.*



Nest of squares patterns. A *nest of squares* is an interesting pattern. Shade one square one color. With another crayon, color the squares around it.



Let a *nest of squares* grow out of the corner.

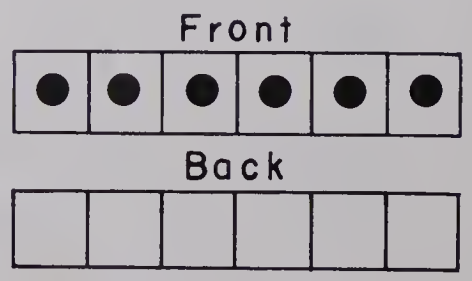


Later, this last pattern will help reveal the connection between *odd numbers* and *square numbers.* Each layer is composed of the next odd number of blocks—1, 3, 5, 7, 9, etc. Each successive square is composed of the next square number—1, 4, 9, 16, 25, etc. The pattern suggests the following table:

$1 = 1$
$4 = 1 + 3$
$9 = 1 + 3 + 5$
$16 = 1 + 3 + 5 + 7$
$25 = 1 + 3 + 5 + 7 + 9$

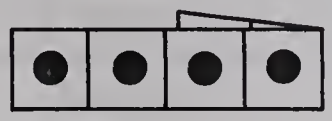
13. Counting activities

A most helpful teaching aid can be made easily out of paper, cardboard, or oilcloth. Establish a convenient unit, about three inches, more or less, and cut strips 2, 3, 4, 5, 6, 7, 8, 9 and 10 units long. Divide each into unit square boxes and put a large solid black circle in each unit square on one side. On the other side, mark off the unit squares, but leave them empty. The strip six units long will look like this:



Fold along each line that separates each strip into boxes so it can be easily folded along any divider. Show a strip to the class. "How many dots do you see?" Show them the back.

Now fold over 1, 2 or 3 dots. "How many do you see now? How many will you see if I turn it around?

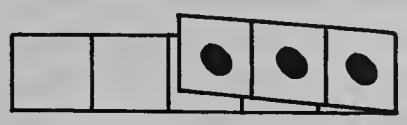


Turn it around so it looks like this:

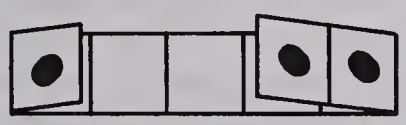
"How many will you see if we straighten out the strip?" (Six.)

Take another strip; fold it before the children see it, and show them the side we will call the back.

"How many dots are on the side you can't see?"



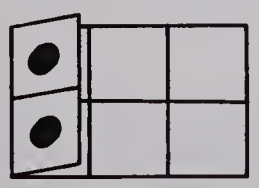
There will be five. "How many are there all together?" There will be eight. Straighten it out and count them. Then fold it differently:



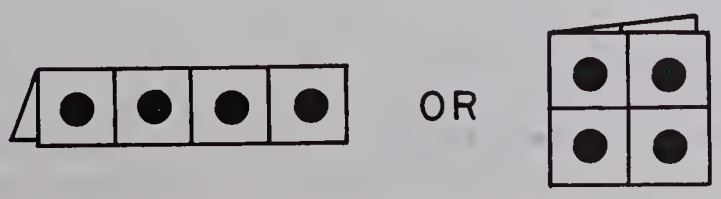
How many do you see? How many are on the other side? How many will you see if we straighten it out so you can see all the dots?"

Let different children fold the strips and display them, asking the questions. You may want to develop games with competing teams.

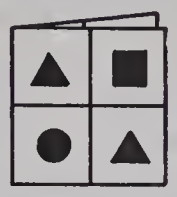
Make another set of strips, two dots deep. An example of such a piece folded might look like:



"How many dots do you see? How many are there on the other side? How many if you can see them all?" Or, it might have been folded thus:



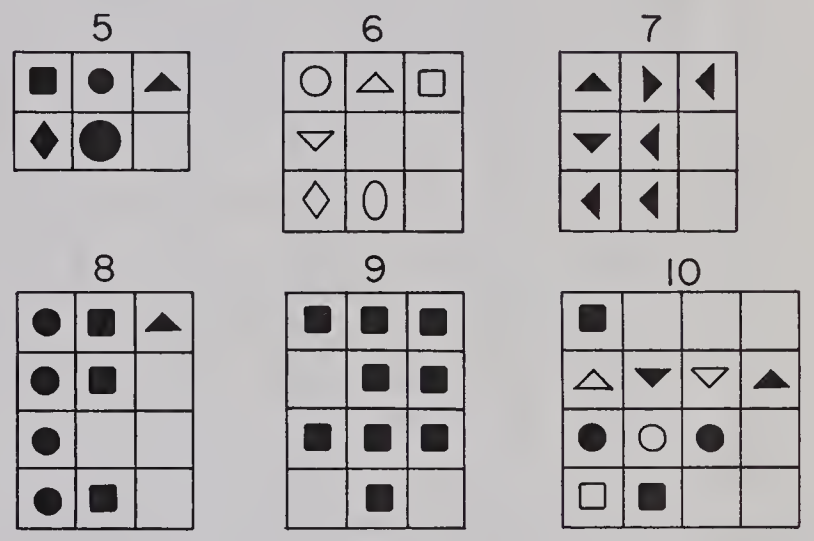
You may prefer using a variety of shapes in place of the dots:



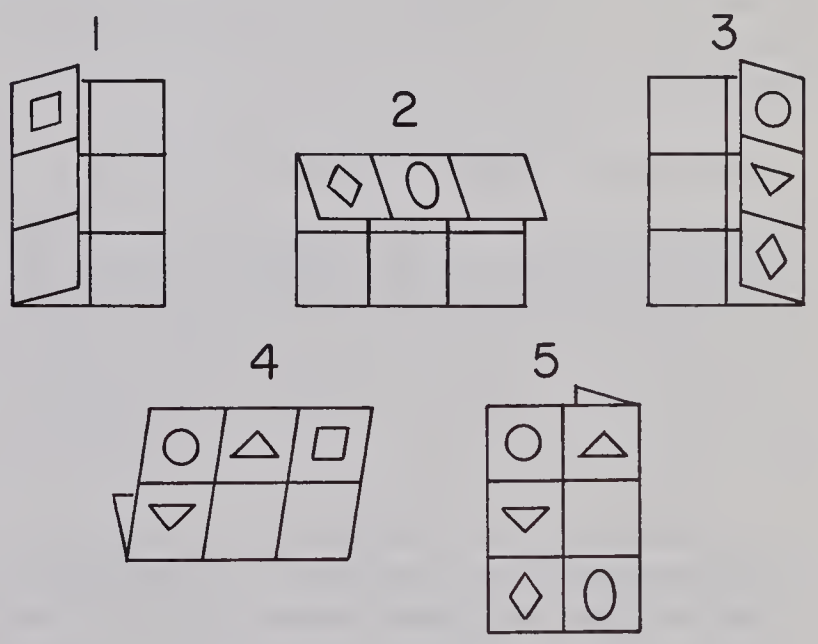
These two sets of strips and rectangular arrays involve an element of mystery, at first. "What's on the side you can't see? How many will there be when you can see them all?" As soon as the element of mystery wears off, many of the facts of arithmetic are ready in the children's minds for immediate recall.

14. Another aid for counting games

Some examples of another useful set of teaching aids are given below.



Each one of these is carefully designed so that it can be folded to display any number of the shapes desired. For example, the piece with six shapes can be folded as follows:



And, without folding, six shapes can be seen. Since each display can be folded to suggest every number from one to the total number of shapes, these few

pieces can be used to suggest every facet of addition and subtraction in which the addend is five through ten. The idea can be extended in both directions.

Show the unfolded card. "How many shapes do you see? Count them. Does everyone agree that there are nine?" Fold the card. "Now, how many do you see? Does everyone agree that we can see only four? How many will you see if I turn the folded card around? How many did you all see before I folded the card? Nine. And you see four now. How many shapes did I hide from you?"

When general agreement has been reached, turn the piece over to check the results. Appoint record keepers to keep track of each example.

NUMBER OF SHAPES

WHOLE SHEET	THE SIDE WE SEE AFTER FOLDING	THE SIDE WE CAN'T SEE
9	4	5
9	3	6
9	1	8
9	7	2

This set of aids is quite versatile. Games can be built around the cards. Small groups of children can use them, individuals taking turns as folder, record keeper and question answerer. Still another dimension can be provided in the activity. "How many empty squares are there? How many with shapes? How many altogether?" Fold the card. "How many empty squares do you see? How many did I fold over?"

And the record grid gets bigger:

BEFORE FOLOING		AFTER FOLOING					
TOTAL SQUARES	TOTAL SHAPES	SHAPES WE SEE	SHAPES WE DON'T SEE	EMPTY SQUARES WE SEE	EMPTY SQUARES WE OON'T SEE	SQUARES WE SEE	SQUARES WE DON'T SEE
12	9	4	5	2	1	6	3
12	9	3	6	1	2	4	5
12	9	1	8	2	1	3	9
12	9	7	2	1	2	8	4

15. Counting and measurement

Pupils must learn the fundamental difference between *counting and measurement.* There are two variables which make the results of any measurement an *approximation:* (1) selection of the unit of measure; and (2) observation in comparing the unit with the object to be measured. All time spent on making these facts clear will be amply rewarded by a more thorough understanding of the problems involved in using the exact *language of counting* as the *language of measurement.*

In such questions as, "How many hands wide is your desk?" whose hand is meant? Is the thumb to be included? Is the hand to be squeezed together? What if the measurement doesn't come out even? Whose opinion is to be accepted?

Or in such a question as, "How many glasses in a quart?" what size glass is to be used? How full shall the glass be? What about spilling?

The purpose of having children participate in measurement activities is to establish for them the idea of: (1) selecting a unit, and (2) observing and recording an approximate comparison.

Explain to the class that in order for them to find out how many "shoe lengths" wide the room is, (stepped off toe-to-heel-to-toe) they first will have to decide whose shoes should be used to do the measuring. Children can match shoe lengths, and select whose shoes will be used in the measuring. Pupils with "approximately" the same length shoe can check the results.

Measurement is also used in "comparing" things that cannot be placed side-by-side. For example, "Which dimension of the room is biggest?" "Is the desk higher than it is wide?" "Are two first-grade rooms about the same size?" "Are two doors about the same height?" "Who lives closest to school?" "Are all city blocks the same length?"

16. Numbers have many names

On the board, draw the following chart:

"Numbers have many names!"				
other names for 3+2	other names for 10	other names for 11-8	other names for 7+5	other names for 9-9

If you draw an odd number of columns for *other names* on the board, select an even number of teams, preferably two. A member of the first team must write on the board *another name for 3 + 2* under the proper column. A member of the next team must write *another name for 10,* and so on.

The teams should alternate each time across. The team that had 3 + 2, 11 − 8, and 9 + 9 the first time around will have 10 and 7 + 5 the next time. Each team scores whenever one of its members give a

correct other name. Whenever one of its members gives an *incorrect other name,* the opposing team scores, provided one of its members spots the error.

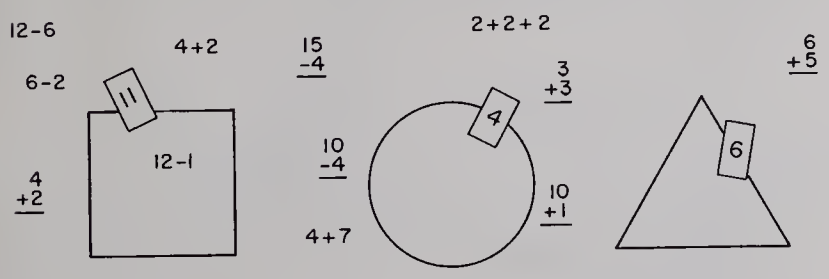
Games such as this let each child participate successfully at his own level. But most important, they drive home the fundamental concept that *numbers have many names.* This is the idea behind all *regrouping.* We can treat 32 as 20 + 12, so that 32 − 8 might be regrouped to read 20 + 12 − 8, and 20 + 4, and finally, 24.

Activities based on the idea that *numbers have many names* help the teacher to estimate the ability of each pupil. They help the teacher to discover who in the class is slow or quick, timid or confident. For example, Miss or Mr. *Way Out Ahead* may reveal themselves by using 3000 ÷ 1000 or $\sqrt{9}$ as other names for 11 − 8. (It is always possible to give the more apt pupils an opportunity to show off their prowess by asking them to write on the board the most difficult looking names for a given number that they can think of.)

A variation can be used to make the board game more difficult. A member of one team starts a name for the number in question, and the other team must complete it. Another name for 3 − 2 might be started as "0 + _" or "19 − _." The next player has to complete the name: "0 + 1," or "0 + 10 − 9," or "0 + 1 × 3 − 2," etc.

Sometimes, *you* may want to begin the other names in order to match the challenge of the situation to the recognized ability of the next player. This is important to the pupil: to be asked a question that is not too easy nor too hard, a question he can answer if he makes an effort.

A second activity. Draw this on the board:



As each expression is discussed, cross it out and rewrite it in the appropriate shape. Be as certain as possible that everyone understands what he is expected to do. Only if the child has no questions about the activity will you be able to judge successfully whether or not he is familiar with the arithmetic involved.

A third activity. Next, have class members tackle the following problem: "How many names for each of the numbers from 0 through 10, can you write in the form of ____ + ____ = ____?"

0	1	2
0 + 0 = 0	0 + 1 = 1	2 + 0
1 name	1 + 0 = 1	1 + 1
	2 names	0 + 2
		3 names

3	4	5

Then, have them record the results in a chart.

Number	Names in form of ____ + ____ = ____	
0	1	
1	2	
2	3	
3	4	
4	5	
5	6	
6	7	
7	8	
8	9	
9	10	
10	11	

This leads to a nice pattern. It suggests that there will be 21 names for 20 in the form of ____ + ____ = ____. Test the pattern. Does it keep on giving you reliable results?

A fourth activity. Next, you should explore the same question for names that are in the form ____ + ____ + ____ = ____.

0	1	2
0+0+0 = 0	0+0+1 = 1	0+0+2 = 2
(1 name)	0+1+0 = 1	0+2+0 = 2
	1+0+0 = 1	2+0+0 = 2
	(3 names)	1+1+0 = 2
		1+0+1 = 2
		0+1+1 = 2
		(6 names)

3	4	5
0+0+3 = 3	0+0+4 = 4	0+0+5 = 5
0+3+0 = 3	0+4+0	0+5+0
3+0+0 = 3	4+0+0	5+0+0
0+1+2 = 3	0+1+3	0+1+4
1+0+2 = 3	0+3+1	1+0+4
0+2+1 = 3	1+0+3	0+4+1
1+2+0 = 3	1+3+0	1+4+0
2+1+0 = 3	3+0+1	4+1+0
2+0+1 = 3	3+1+0	4+0+1
1+1+1 = 3	2+2+0	0+2+3
(10 names)	2+0+2	2+0+3
	0+2+2	0+3+2
	1+1+2	2+3+0
	1+2+1	3+0+2
	2+1+1	3+2+0
	(15 names)	3+1+1
		1+3+1
		1+1+3
		2+2+1
		2+1+2
		1+2+2
		(21 names)

4 → 3+1
2+2
2+1+1
1+1+1+1
(4 names)

5 → 4+1
3+2
3+1+1
2+2+1
2+1+1+1
1+1+1+1+1
(6 names)

6 → 5+1
4+2
4+1+1
3+3
3+2+1
3+1+1+1
2+2+2
2+2+1+1
2+1+1+1+1
1+1+1+1+1+1
(10 names)

7 → 6+1
5+2
5+1+1
4+3
4+2+1
4+1+1+1
3+3+1
3+2+2
3+2+1+1
3+1+1+1+1
2+2+2+1
2+2+1+1+1
2+1+1+1+1+1
1+1+1+1+1+1+1
(14 names)

8 → (21 names)

9 → (29 names)

10 → (39 names)

The results, once more, are to be recorded in a chart.

Numbers	Names in form of ___+___+___=___	
0	1	
1	3	
2	6	
3	10	
4	15	
5	21	
6	?	

Pupils will probably discover for themselves that it is helpful to devise a system for writing down the different names of each number. Otherwise it becomes difficult to keep track of the combinations. (Note: do not be disturbed if you cannot predict the number of combinations possible for 11. It took two famous mathematicians a long time to develop a formula that allows such a prediction to be made, and the formula is a complicated one.)

A fifth activity. Each of the following problems presents a pattern of its own:

1) "How many names for each of the numbers 1 through 10 can you find, using any one pair in one order only and omitting 0?"

0 → none

1 → none

2 → 1+1 (1 name)

3 → 2+1 (1+2 does not count, because 2+1 has already been included)
1+1+1
(2 names)

4 → 3+1
2+2
2+1+1
1+1+1+1
(4 names)

5 → 4+1
3+2
3+1+1
2+2+1
2+1+1+1
1+1+1+1+1
(6 names)

6 → 5+1
4+2
4+1+1
3+3
3+2+1
3+1+1+1
2+2+2
2+2+1+1
2+1+1+1+1
1+1+1+1+1+1
(10 names)

7 → 6+1
5+2
5+1+1
4+3
4+2+1
4+1+1+1
3+3+1
3+2+2
3+2+1+1
3+1+1+1+1
2+2+2+1
2+2+1+1+1
2+1+1+1+1+1
1+1+1+1+1+1+1
(14 names)

8 → 7+1
6+2
6+1+1
5+3
5+2+1
5+1+1+1
4+4
4+3+1
4+2+2
4+2+1+1
4+1+1+1+1
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3+1+1+1+1+1
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2+2+2+1+1
2+2+1+1+1+1
2+1+1+1+1+1+1
1+1+1+1+1+1+1+1
(21 names)

9 → 8+1
7+2
7+1+1
6+3
6+2+1
6+1+1+1
5+4
5+3+1
5+2+2
5+2+1+1
5+1+1+1+1
4+4+1
4+3+2
4+3+1+1
4+2+2+1
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2+2+2+1+1+1
2+2+1+1+1+1+1
2+1+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1
(29 names)

10 → 9+1
8+2
8+1+1
7+3
7+2+1
7+1+1+1
6+4
6+3+1
6+2+2
6+2+1+1
6+1+1+1+1
5+5
5+4+1
5+3+2
5+3+1+1
5+2+2+1
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2+2+2+2+2
2+2+2+2+1+1
2+2+2+1+1+1+1
2+2+1+1+1+1+1+1
2+1+1+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1+1
(39 names)

2) "Find all the names for the *even* numbers 1 through 10, using the sum of *even* numbers only and omitting 0:

4 → 2+2 (1 name)

6 → 4+2
2+2+2 (2 names)

8 → 6+2
4+4
4+2+2 (4 names)
2+2+2+2

10 → 8+2
6+4
6+2+2
4+4+2
4+2+2+2
2+2+2+2+2
(6 names)

3) Find the names for all the numbers 2 through 10, using the sum of *odd* numbers only, and omitting 0:

2 → 1+1 (1 name)

8 → 7+1
5+3

(more, next page)

3 → 1+1+1 (1 name)
4 → 3+1
1+1+1+1 (2 names)

5 → 3+1+1
1+1+1+1+1(2 names)

6 → 5+1
3+3 (4 names)
3+1+1+1
1+1+1+1+1+1

7 → 5+1+1
3+3+1 (4 names)
3+1+1+1+1
1+1+1+1+1+1+1

9 → 7+1+1
5+3+1
5+1+1+1+1
3+3+3 (7 names)
3+3+1+1+1
3+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1

	Names
2	1
3	1
4	2
5	2
6	4
7	4
8	6
9	7
10	10

17. Names for numbers in different forms. How many names could be written for zero in the next green box above 1, there are two possibilities. In completing this row of green boxes, a familiar pattern becomes apparent: one name for 0; two names for 1; three names for 2; four names for 3; five names for 4.

As observed earlier, a very different condition develops when the form becomes: "__ − __." Now there are unlimited possibilities.

Below the number line, the form "__ × __" appears. The pattern here is quite different: There are an unlimited number of names for 0 in the form "__ × __"; one name for 1 in the form "__ × __"; two names for 2 in the form "__ × __"; two names for 3 in the form "__ × __"; three names for 4 in the form "__ × __."

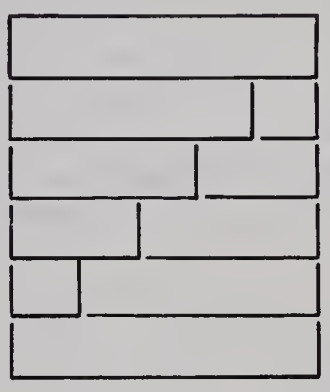
In this table, when there are only two names for a number in the form of "__ × __," the number is a prime number (a number that is divisable by exactly 2 different numbers, itself and 1): 2, 3, 5, 7, 11, 13, etc.

Finally, we discover that the form of "__ ÷ __" yields an unlimited group of names for each number. While there are three ways given here of writing these names, our main interest is in the form which leads so naturally into fractions: $\frac{1}{1}$, $\frac{6}{3}$, $\frac{3}{1}$, etc.

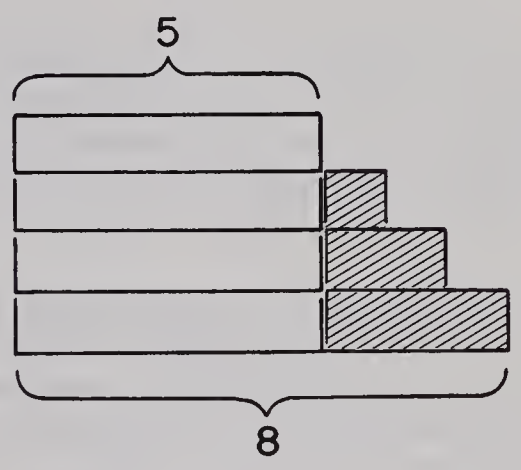
Care should be taken with other names for 0 in the form of division. It has to be asserted that division by 0 doesn't make any sense in arithmetic; it must be excluded. The only permissible names are 0 divided by any number other than 0: $\frac{0}{2}$, $\frac{0}{1}$, $\frac{0}{3}$, $\frac{0}{4}$, etc.

Manipulative materials. Give each student a series of paper or cardboard strips which he can manipulate. (A 1-inch square for one, a 1 × 2 strip for two, a 1 × 3 strip for three, etc.) Blocks or rods in a series of lengths representing 1 to 10, which you may already have in your classroom, will serve the same purpose. These can be used in a variety of ways, not only to reconstruct the sizes and relationships on these pages, but also to work out such problems as, "How many names has a number?"

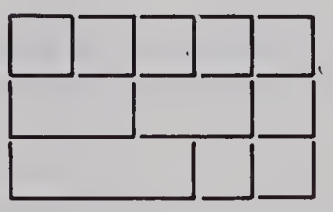
For example, pupils can make "little pictures" of 5 in such ways as:



5 + 0 = 5
4 + 1 = 5
3 + 2 = 5
2 + 3 = 5
1 + 4 = 5
0 + 5 = 5



5 - 0 = 5
6 - 1 = 5
7 - 2 = 5
8 - 3 = 5



1 + 1 + 1 + 1 + 1 = 5
2 + 2 + 1 = 5
3 + 1 + 1 = 5
etc.

Fractions. "Is two dollars divided by two the same as one dollar?" ($\frac{2}{2} \rightarrow 1$) "If we divide 4 sandwiches by 2, do we have two sandwiches? Are 6 sandwiches divided by 3, two sandwiches?" ($\frac{4}{2} \rightarrow \frac{6}{3}$) "Would one sandwich divided by two be the same as 3 sandwiches divided by 6, or 2 sandwiches divided by 4?" ($\frac{1}{2} \rightarrow \frac{3}{6} \rightarrow \frac{2}{4}$)

If the dime is used as a basic unit, a nickel is to be thought of as $\frac{1}{2}$ a dime, a quarter as $2\frac{1}{2}$ dimes, a half dollar as 5 dimes, fifteen cents as $1\frac{1}{2}$ dimes. Continuing with the dime as the basic unit, three nickels would be called $3\frac{1}{2}$ dimes, and a quarter and a nickel would be $2\frac{1}{2}$ dimes and another $\frac{1}{2}$ dime."

In terms of dimes, the following could be accurately interpreted: $\frac{2}{2} = 1$; $\frac{1}{2} + 2\frac{1}{2} = 3$; $\frac{3}{2} = 1\frac{1}{2}$; $2\frac{1}{2} = \frac{5}{2}$.

Each time pupils answer questions like these, the mathematical symbol should be written on the board. The arrow, to be read "suggests" or "is the same idea as," is preferred in situations like this, to "=" ("equals"). The idea has not yet been firmly established that fractions are actually numbers or can be treated as numbers. Pupils do not know that they can add, subtract, multiply, and divide fractions. What pupils have been learning is that $\frac{4}{2} = \frac{6}{3}$; and that these are two names for the same number.

18. Row and column activities

The game, "Row-Column," can be adapted to any level of maturity. Use a "magic square," for example.

	A	B	C
X	8	1	6
Y	3	5	7
Z	4	9	2

A player for "Row" chooses X, Y or Z, and writes it down secretly. Then a "Column" player selects A, B or C. The players announce their choices. The number written where the announced row and column cross becomes "Row's" score.

Next a player for "Column" starts by choosing A, B, or C and secretly writing it down. Then a "Row" player selects X, Y, or Z. "Column's" score is discovered by finding the number where the choices of the two players cross.

Each person or team keeps a cumulative score. The first to reach a predetermined total, such as 25 or 50, wins.

Other grids can be made by adding, subtracting or multiplying the number in each cell of the magic square by one constant number:

Add 2

10	3	8
5	7	9
6	11	4

Subtract 1

7	0	5
2	4	6
3	8	1

Multiply by 2

16	2	12
6	10	14
8	18	4

In the examples above, the numbers in the rows and columns have been selected so that the sum for any three numbers down or across is the same. Neither "Row" nor "Column" is favored; chance for a high score is even. You may use smaller numbers on the square:

	A	B	C
X	1	2	1
Y	0	1	3
Z	3	1	0

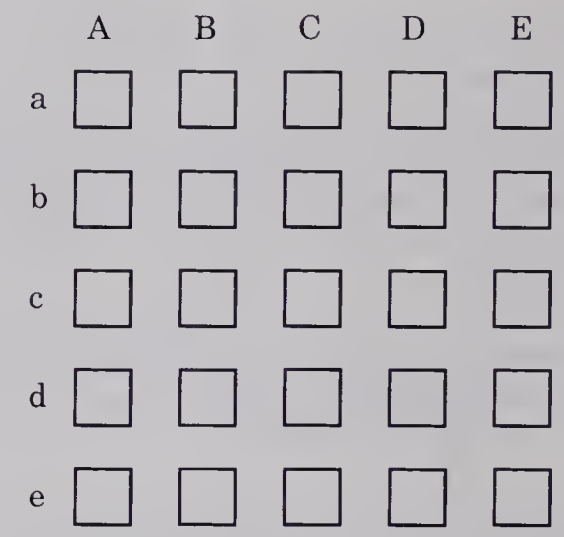
	D	E	F
U	0	1	2
V	3	0	0
W	0	2	1

	G	H	I
R	1	2	3
S	5	0	1
T	0	4	2

19. Exploring the magic square

More capable children may want to explore the principle of the magic square. In the first magic square shown above, the rows, columns and principal diagonals each total 15. If 6 were added to each cell, the new magic number would be 33; if each cell were multiplied by 3, the magic number would be 45.

Using notations. Put a diagram such as the following, on the board:



The notations B (AB,a) would refer to the number of boxes in colums *A* and *B* that are also in row *a* or, 2. A diagram such as the one above might represent seats in a classroom.

As soon as a pupil understands the idea clearly, he can look at a notation such as, N (ABD,ce) and know that the number N stands for is 6, because an array of 3 columns and 2 rows includes 6 items.

N(BC,ab) + N(D, abc) = 7 because 4 + 3 = 7. Later on, this can be interpreted as $(2 \times 2) + (1 \times 3) = 7$. Please do not suggest to pupils that this is an example of algebra. It is nothing of the kind. It is simply a kind of shorthand used in talking about a picture, such as the one on the board. Algebra is also a shorthand, but it is a different kind of shorthand.

Whenever you can, use notations in talking about a picture, an example, or a problem. Just remember that any notation is of use only when an agreement has been made about the meaning of the notation. Such notations (or shorthand signs) as "6," "10," "+," "×," "=," etc. are very useful because there is general agreement about what they mean in an arithmetical context.

More grids. Using any letters you want, create a grid similar to the following:

	a	b	c	d	e	f	g
h	B	B	B	E	E	E	E
i	B	B	B	E	E	E	E
j	F	F	F	E	E	E	E
k	F	F	F	P	P	P	P
l	F	F	F	P	P	P	P
m	F	F	F	P	P	P	P
n	F	F	F	P	P	P	P

According to the above grid, "N (ab,ijk)" means that the number of letters in columns *a* and *b,* and in rows *i, j* and *k* is 6. (There are 2 B's and 4F's.) A chart, based on the above grid, could be drawn:

	Total number of letters	Number			
		B's	E's	F's	P's
N (ab,ijk)→	6	2	0	4	0
N (cde,h)→	3	1	2	0	0
N (fg,mn)→	4	0	0	0	4
N (bcd,lmn)→	9	0	0	6	3

To vary the exercise, fill in the five columns with appropriate figures and ask pupils to give a notation that could be written in the left-hand column. For example:

?	8	2	4	2	0
?	4	1	1	1	1

Additional activities can be given to the children that finish the page first. This frees the teacher for individual work. Another grid that might be used is of shapes:

	a	b	c	d	e	f
g	□	□	□	△	△	△
h	□	□	□	△	△	△
i	•	•	•	△	△	△
j	○	○	○	○	○	○
k	○	○	○	○	○	○

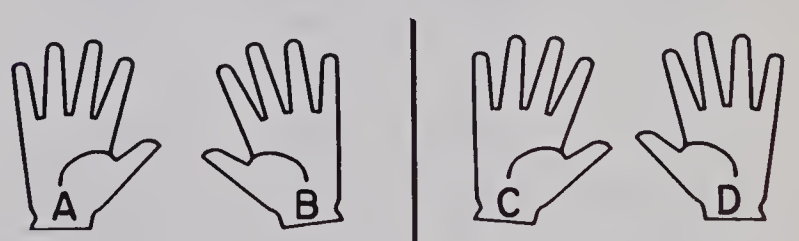
20. Multiplication activities

The series of activities that follows could be repeated in many different ways. Select the examples you like best, and repeat them often. Through these experiences, pupils will gain a real understanding of "multiplication."

a. Pairing digits. Put the chart below on the board. Fill in the "Digit on Left" and "Digit on Right" columns, as you go. Pupils should give you the answers to put in the other columns.

Digit on Left	Digit on Right		Total
2	0,3	20, 23	2
1, 2	1, 2	11, 12, 22, 21	4
1, 2, 3	5	15, 25, 35	3
7, 8, 9	1, 2	71, 72, 81, 82, 91, 92	6
1, 2, 3,	4, 5, 6	14, 15, 16, 24, 25, 26,	9
		34, 35, 36	

b. Pairing gloves. State the problem in this way: "Mary's mother bought her two pairs of gloves that were alike. In how many ways can she put them on?"



AC, AD, BC, BD

"If Mary lost one, in how many ways can she make a pair of gloves? Suppose she lost B?" (AC and CD.)

"Suppose she lost C, as well as B?" (AD only.)

"Suppose she found B and lost A as well as C?" (None—for she would have two gloves for the same hand.)

c. Couples. Here is another problem statement: "There were three men and three women sitting together. Jim said his parents were in the group. He wanted John to guess which couple his parents were. One way for John to find out would be to try all the possible combinations. But to do this he would have to give the men and women letter names in order to make a chart of the possibilities. The chart would look like this:

Men	*Women*
Mr. *A*, Mr. *B*, Mr. *C*	Mrs. *X*, Mrs. *Y*, Mrs. *Z*

"The possible combinations would be: *AX, AY, AZ, BX, BY, BZ, CX, CY, CZ*

"If John knew that Mr. *A* was not Jim's father, how many possibilities are there? (6-*BX*, *BY*, *BZ*, *CX*, *CY*, *CZ*.)

"If John knew that Mrs. *X* was not Jim's mother, how many possibilities are there? (4-*BY*, *BZ*, *CY*, *CZ*.)

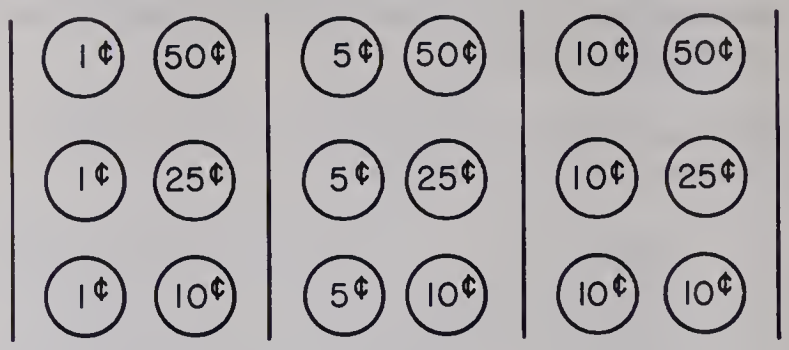
"If Jim said, 'Mr. *C* is not my father,' who would his father be?" (Mr. *B*.)

"If Mrs. *Y* is not Jim's mother, who is his mother?" (Mrs. *Z*.) "Who are Jim's parents?" (Mr. *B* and Mrs. *Z*.)

d. Pairing coins. You need two boxes and some coins. In one box put one penny, one nickel, and one dime. In another, put one dime, one quarter, and one half dollar. Have individuals from the class close their eyes and pick coins from the boxes. Make a record of the coins and total amount of money each child has picked:

Name	Box A	Box B	Total
Mary	1c	10c	11c
John	5c	25c	30c
Bill	10c	50c	60c
Barbara	5c	10c	15c
Alex	10c	10c	20c
Jean	5c	50c	55c
Bob	1c	50c	51c
William	10c	25c	35c
Helen	1c	25c	26c

The above chart includes the nine combinations that are possible. After you have recorded the first nine draws, ask if there are any repetitions. If there are, eliminate them and continue as before. When the class has come up with six or seven unique combinations, ask, "Can you think of any other possible combinations that we haven't had yet?" Pupils may find it easier to find combinations if they make a systematic record of the possibilities:



Repeat this same activity with fewer than three coins. Keep reducing or changing the number of coins—2 and 3, 3 and 2, 2 and 2, 1 and 2, 2 and 1, 1 and 1, 0 and 3, 2 and 0. When a record (table) is finished, erase the amounts in column A and B. Ask, "Can we tell which coins each child drew if we know only the total amount?" (Of course!)

e. How many combinations? This time, you must—Take three boxes. Place a penny and nickel in A; a nickel and dime in B; a dime, quarter, and half dollar in C. Have pupils draw a coin from each box. There are 12 possible combinations (because $2 \times 2 \times 3 = 12$) and the total amount represented by each draw will be different or unique. The following chart will result (do not include the right-hand column until the idea of multiplication has emerged):

A	B	C	Combinations	Because
2	2	3	12	$2 \times 2 \times 3 = 12$
2	2	2	8	$2 \times 2 \times 2 = 8$
2	1	3	6	$2 \times 1 \times 3 = 6$
2	1	1	2	$2 \times 1 \times 1 = 2$
1	2	1	2	$1 \times 2 \times 1 = 2$
1	1	3	3	$1 \times 1 \times 3 = 3$

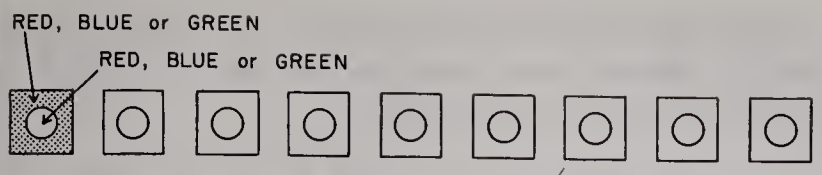
If you use coins in the pattern suggested, every amount will be unique, so the selections can be reconstructed from a column showing only the amounts.

f. How many ways to color? The question here is: How many ways are there to color the design below, if the following conditions are met:

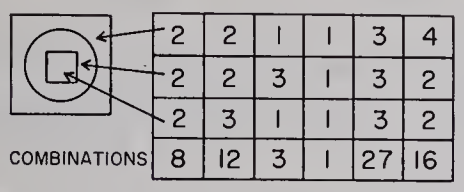
A—The area between the square and circle is to be red, blue or green.

B—The area inside the circle is to be red, blue or green.

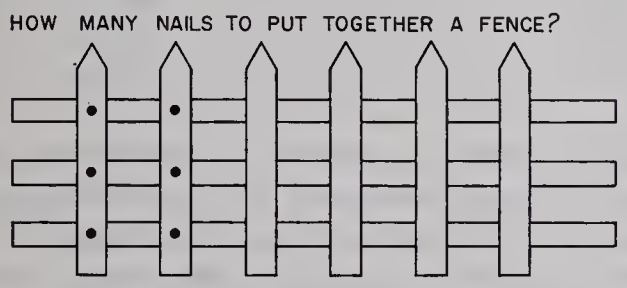
C—Both areas must be colored.



There are nine ways (because $3 \times 3 = 9$). Repeat the activity using three colors and two colors (result —6 combinations). To make the activity more difficult, increase the number of colors: three colors and four colors, for example, which produce 12 possible combinations. This activity can be extended to a design having three areas:



g. How many nails? Here is a seventh activity: Ask the class to tell you how many nails it is going to take to put together a fence like this:



Cross Pieces	Uprights	Nails
3	2	6
3	3	9
3	4	12
3	5	15
3	6	18

h. How many combinations of money? And an eighth—This example is planned to help develop the idea of multiplication and, at the same time, to provide pupils with an exercise in writing amounts of money. Ask pupils how many different amounts of money could be made by combining one each of the following paper money and coins:

Bills for $1.00, $5.00, and $10.00 and coins for 1c, 5c, 10c, 25c, 50c (Answer: 15, because 3×5=15)

$1.01	$1.05	$1.10	$1.25	$1.50
$5.01	$5.05	$5.10	$5.25	$5.50
$10.01	$10.05	$10.10	$10.25	$10.50

Bills for $1.00, $5.00, and $10.00 and coins for 5c, 10c, 50c (Answer: 9, because $3 \times 3 = 9$)

Try examples in which there is either no bill or no coin. In such cases there can be no combinations. This should serve to suggest the idea that $0 \times 3 = 0$, $2 \times 0 = 0$, etc.

i. How many batteries? From the world of sports—A baseball team has four pitchers and four catchers. How many different pitcher-catcher combinations can they plan on using? (A pitcher-catcher combination is called a "battery." Another team has four pitchers, and three catchers. How many combinations *less* than the first team can they make? (Four less—$4 \times 4 = 16$, $4 \times 3 = 12$ and $16 - 12 = 4$.) The chart below records the number of possible combinations that can be made if the number of pitchers is from 0 – 4 and the number of catchers if from 0 – 4.

	Pitchers				
Catchers	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	6	8
3	0	3	6	9	12
4	0	4	8	12	16

j. How many combinations of fingers? The question: "In how many ways can you combine a finger of your right hand with a finger of your left hand?"

Make the grid shown below. Ask a pupil to make a combination. Record it. Next, ask a child to make a different one. Record that. Etc.

	Right Hand				
Left Hand	Thumb	Fore	Middle	Ring	Little
Thumb					
Fore					
Middle					
Ring					
Little					

There will, of course be 25 combinations.

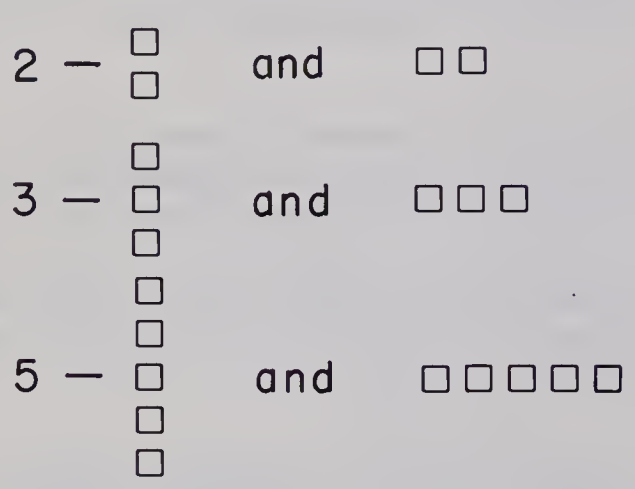
"What happens if we don't count thumbs?" (Only 16 combinations.) "Suppose we don't count thumbs or little fingers?" (Only 9 combinations.) Whenever a finger is eliminated from *each hand,* the number of possible combinations is still the square number of the fingers on any one hand: 16, 9, 4, 1, 0 (4×4, 3×3, 2×2, 1×1, 0×0).

k. Column and row arrangements. The problem is: Can you arrange 12 chairs in rows and columns so that each row has the same number of chairs and each column has the same number of chairs?

Total Chairs	In each Row	In each Column
12	3	4
12	4	3
12	2	6
12	6	2
12	1	12
12	12	1
16	4	4
16	8	2
16	16	1

If 24 chairs were used, it would be possible to illustrate: 1×24, 24×1, 12×2, 2×12, 8×3, 3×8, 6×4, 4×6. Other illustrations of multiplication can be shown by varying the number of chairs.

"In compliance with the rule that the number of chairs in each row and column be equal, what number of chairs would have to be set up so that there was only 1 row or 1 column?" (2, 3, 5, 7, 11 13, 17, 19, 23, etc.)



"These numbers are different from others. They are called prime numbers." Ask, "Can you find more *prime numbers*?"

l. How many first flights? You are a travel agent: In planning a trip, Mr. Traveler wanted to leave from New York, Boston, or Washington, D.C. He wanted to visit London, Paris, or Rome first. In how many ways could he plan his trip?

	Leaving from:		
Arrive	NewYork	Boston	Washington, D.C.
London	NY - L	B - L	W - L
Paris	NY - P	B - P	W - P
Rome	NY - R	B - R	W - R

Vary the number of cities Mr. Traveler might want to leave from, and the number of cities he might want to visit first.

m. How many combinations of food and drink? A menu:

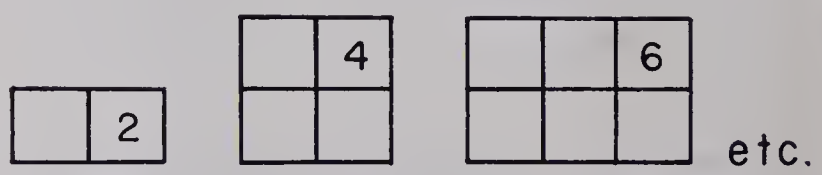
Sandwiches—	Hamburger	25c
	Tuna Fish	19c
	Hot Dog	17c
	Cheese	11c
Beverages—	Regular Milk	10c
	Chocolate Milk	13c
	Soda	6c

Jim had a sandwich and a beverage. His bill was 25c. What did he have? In how many ways could he have ordered a sandwich and a beverage from the same menu (12)?

	Reg. Milk 10c	Choc. Milk 13c	Soda 6c
Hamburger 25c	35c	38c	31c
Tuna Fish 19c	29c	32c	25c
Hot Dog 17c	27c	30c	32c
Cheese 11c	21c	24c	17c

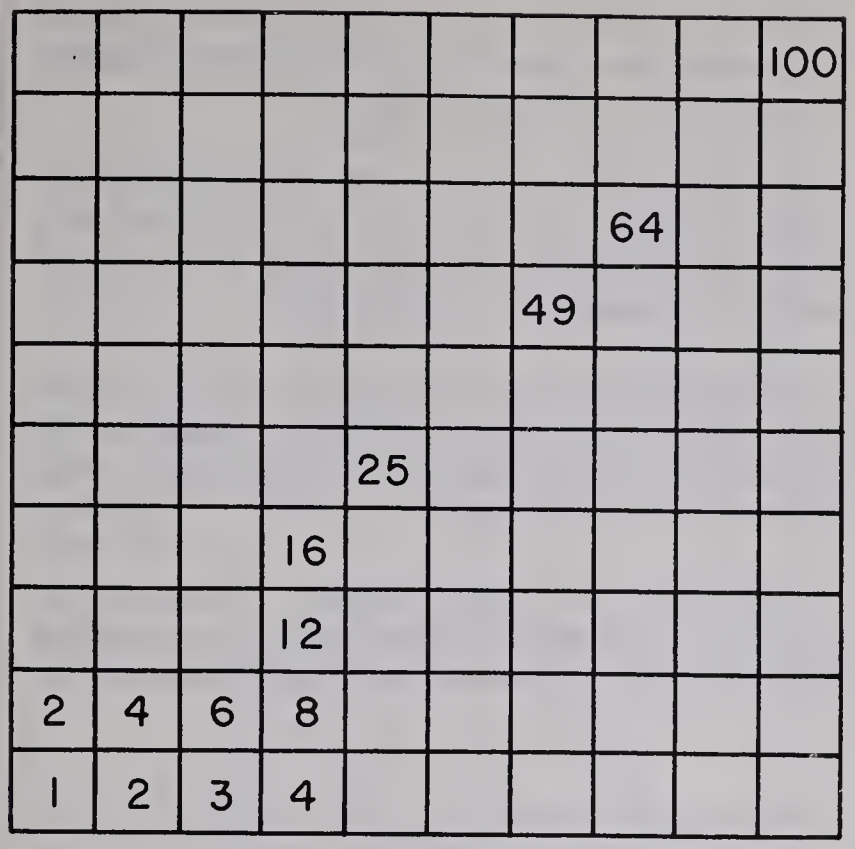
Vary the menu in numbers of items and prices. Be careful not to use prices that result in the same amount for two different combinations. One easy method is to add or multiply all the items by one figure. In this way you preserve the pattern of uniqueness worked into the example given.

n. How many squares? With scissors, you make a machine: From paper that is squared on one side, cut as many rectangles as possible, each containing a number of squares. No side should be more than the length of ten squares.



In the upper right-hand corners, record the number of squares. Arrange them in order of size. Stack them so the bottom left corners are all together—putting the one with the largest number down first, as shown at the top of the next column.

Wherever there is a square without a number, a rectangle can be cut to fill the gap. (You may want to start with all rectangles whose sides are not more than the length of five squares—so that 25 will show in the upper right-hand corner.) You have created what we call a *multiplying machine.*

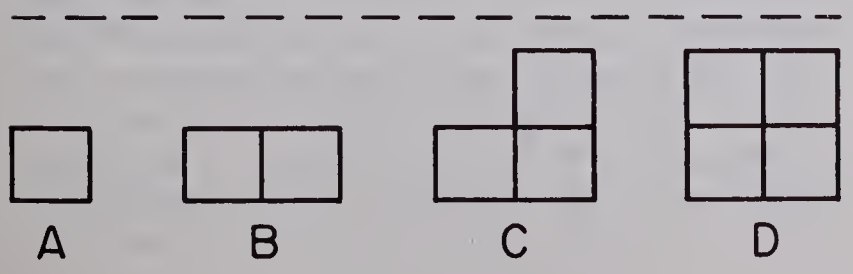


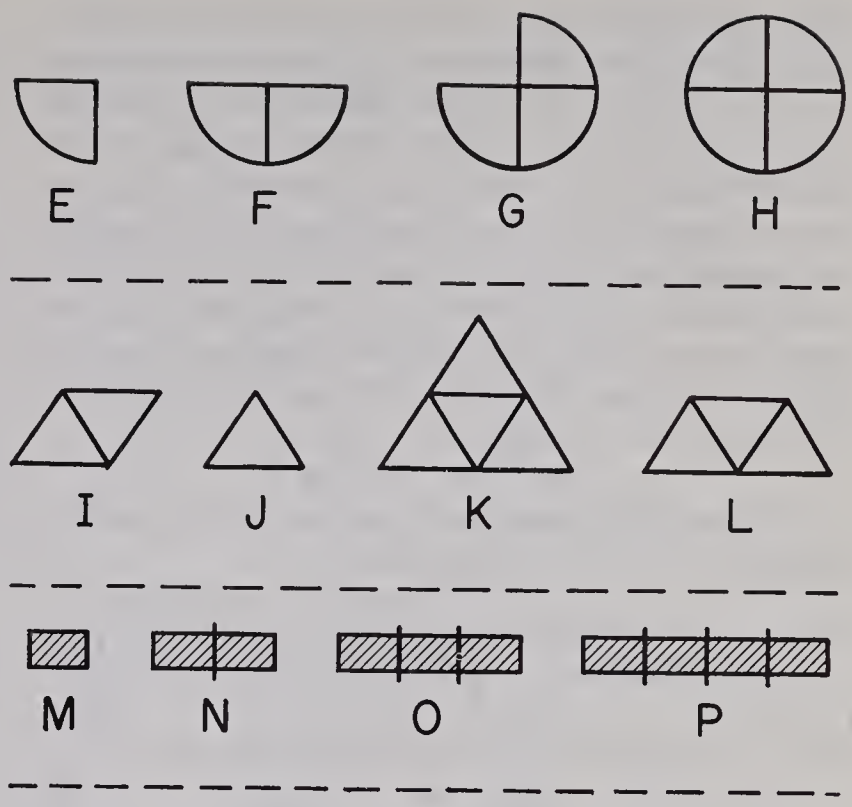
Put the rectangles face down on a table. Ask, "Who can find the rectangle divided into 4 parts? Who can find the 9 squares, 16 squares, 25 squares, etc. What shape are these parts?" (Squares; and again we come upon the square numbers—0, 1, 4, 9, 16, 25, 36, 49, etc.) Teams can play a game. One team says, "Find 24." If the other team finds "24," it belongs to them. If not, the first team gets a chance to find it. Each team collects the rectangles it has correctly picked. Score is kept by adding the *squares* of the rectangles each team has collected.

Each team collects the rectangles it has correctly picked. Score is kept by adding the *squares* of the rectangles each team has collected.

o. Shapes as units in multiplication. We know that "*If* . . . , *then* . . ." is the most used expression in mathematics. "*If* John has ten marbles and loses seven, *then* he must have three left." "*If* I draw a line from a point inside a triangle to a point outside a triangle, then I must cross the triangle." "*If* 19 is more than a first number and that number is more than a second number, *then* 19 is more than the second number." "*If* we agree that the size of a shape such as a certain square suggests five, *then* a shape twice as large suggests 10."

Many similar activities can be developed around diagrams drawn on the board:



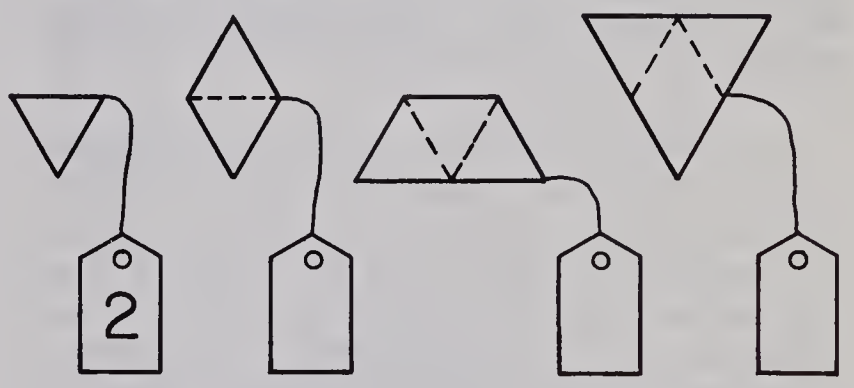


"*If* the size of A suggest five cents, *then* what is suggested by the sizes of B, C, and D?" B could suggest two nickels, one dime, or ten pennies.

"If the size of H suggests a dozen eggs, then what do the sizes of E, F, and G suggest?"

The designs used to reflect relative sizes may be any shape, any order, but not confusing.

Next, draw this diagram on the board:

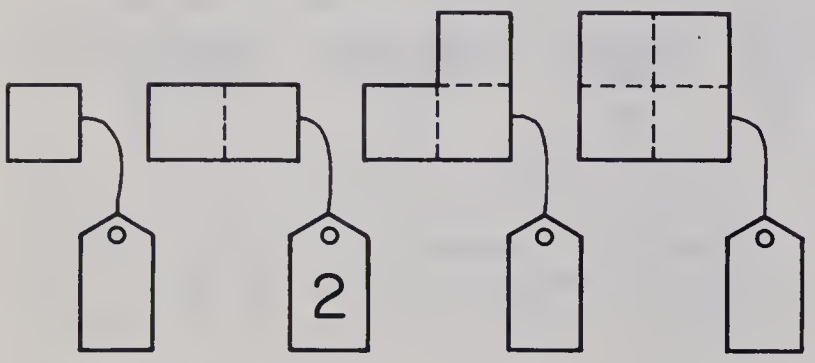


"*If* we say that the size of the first shape suggests *two* as the label indicates, *then* how would we label the other shapes?" The tags would read *2, 4, 6,* and *8*. Next, erase the *2,* the *6,* and the *8,* leaving the *4*. Ask pupils, "Suppose to start with we were given the number 4 in the second tag. How would we number the other tags?" The same system of labeling would result: *2, 4, 6,* and *8*. Repeat the activity, starting with only the tag labeled *6*; then with only the tag labeled *8*.

Next, erase all labels and put *6* in the second tag. "If the tag on the second shape were marked with a 6, *then* how would we label the tags for the other shapes? Is there a shape twice as large as the one we

have labeled with a 6?" (Yes) "What should its label be?" (12) "Now, how do we label the other tags?" (3 and 9) By this time, most of the class will have the idea and will be able to proceed with the page. Avoid any temptation to verbalize the relationship between shapes in some way such as: "The smallest tag should be half of the next one." Either a child has understood the activity, or he has not. If he has, then he can label the pictures properly. If not, he can't label them. There need be no verbalization to determine this question.

If some children are still confused, draw another diagram on the board, such as:



Repeat the development as already outlined. There is a subtlety in this activity that may properly disturb some children. They are used to counting *things*; a dot is a dot, regardless of its size. Therefore, it is necessary to make it clear that the ground rules have been changed. Emphasize the fact that we *are* concerned here with size. Just as it is possible to arrange collections of things according to the number of objects in them, it is also possible to arrange objects according to their size.

We have gone even further than a simple consideration of grouping objects according to size, by introducing the idea of *relative size.* If children have not absorbed this idea from the examples, keep drawing more pictures, more illustrations. The fact is that the child who is distrubed by the change in *rules* may be the most mathematically inclined. In any event, it is wise to accept the responsibility for introducing the new idea by taking great care in making it clear to the pupils.

Another possible approach is, "Suppose we have some pieces of paper of different sizes:

"Here are some small blocks. Now, how many blocks do you need to cover each piece of paper?" Keep a record of the results. Then put the blocks away.

"Suppose we had blocks of all the same size. If two of them would just cover the next-to-the smallest piece of paper, then how many would be required to cover the other pieces of paper?"

Don't by-pass or overlook the slowest child. The ability to make an agreement involving relative sizes is important in later problems of multiplication, area, numeration, fractions, etc.

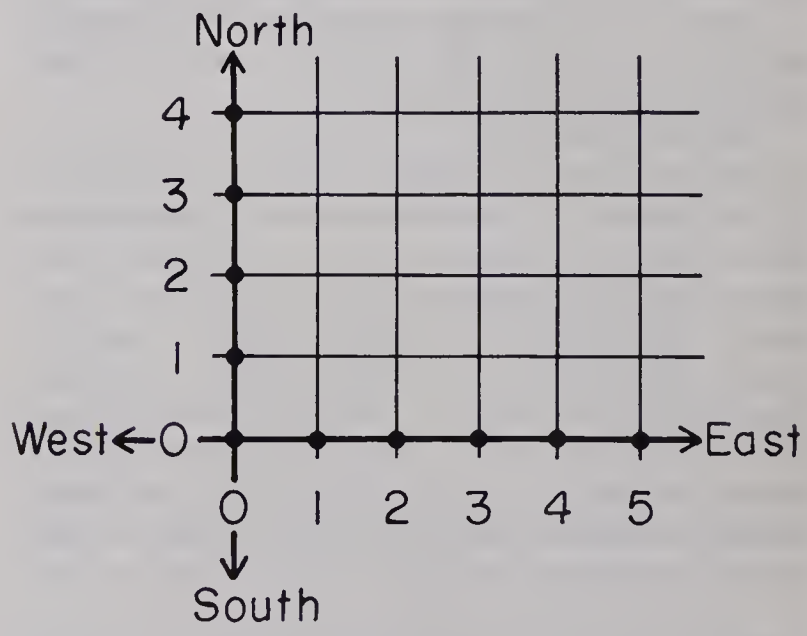
p. Using grids as a multiplication activity. Draw on the board two number lines that intersect, as shown below. Label each line beginning with "0" at the point of intersection.

Ask pupils, "How many jumps is it from 2 on one line to 3 on the other? How many jumps is it from 4 on one line to 4 on the other?" Record the results.

Starting point on one line	2	4	3	2	1	4	3
Landing point on the other	3	4	2	2	4	0	3
Jumps (of 1 each) required	5	8	5	4	5	4	6

Next, draw lines through each point on both number lines and introduce the game, *Taxicab Arithmetic.*

"Let's agree that each line we draw is a street on which taxicabs can travel. Of course, they must stay on the streets, but they can travel in any direction. The distances between intersections are *blocks.*

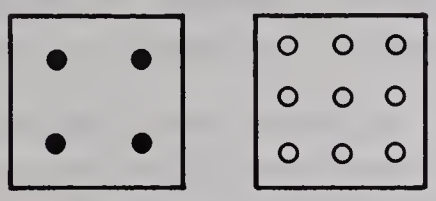


"We will call the West-East number line, W-E *Main Street,* and the South-North number line, S-N *Main Street.* Suppose a taxicab starts at 2 on S-N Main Street; it travels four blocks east and two blocks south. Where does it end up? How many blocks did it travel?" (It ended up at 4 on W-E Main Street, and it went six blocks.) "Could it have gotten there by a shorter route?" (No) "If it had stayed on

one of the Main Streets all the way, would it have traveled more blocks?" (No)

Try many trips from points on one Main Street to points on the other. Always, the shortest way is the same as if the trips were made only on the Main Streets. This game develops the two-dimensional concept that enables us to use the same diagram as a suggestion of multiplication. "Select any corner or intersection *not* on a Main Street. Draw a line from that point to take the shortest trip to each Main Street. How many square blocks are surrounded by those two trips and the two Main Streets?"

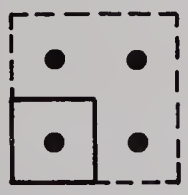
q. Square arrays. Sometimes it is possible to arrange things in a square array. For example:



In the above pictures, four dots have been arranged in two columns and two rows; and nine dots have been arranged in three columns and three rows.

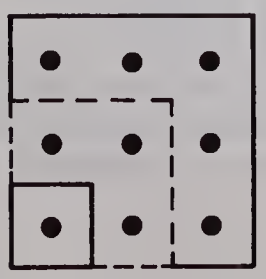
The following pictures:

suggest it is possible to arrange 0 things in 0 rows, and one thing in one row. The question is, "What numbers of things can be arranged in this way?" (0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, etc.) One way to suggest how this particular series of numbers grows is to begin with one row and one column.

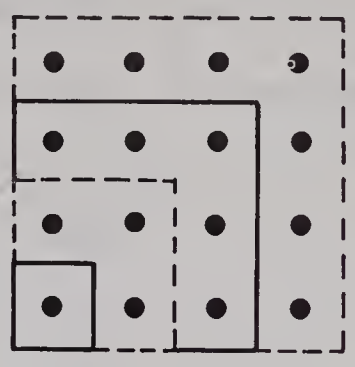


Then arrange two rows and two columns of dots, for example:

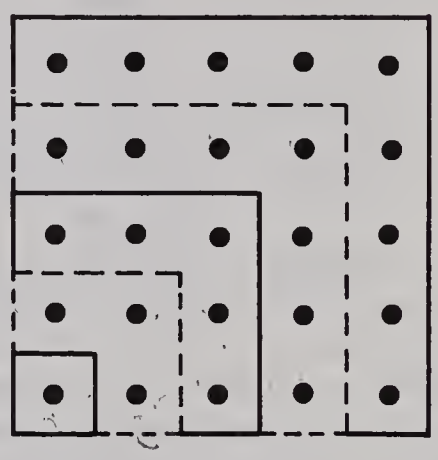
To one dot we have added three dots . . . and $1 + 3 = 4$. Next make three rows and three columns of dots:



To one and three dots we have added five dots . . . and $1 + 3 + 5 = 9$. Continue by making four rows and four columns of dots.



To one and three and five dots we have added seven dots . . . and $1 + 3 + 5 + 7 = 16$. Next:



To one and three and five and seven dots, we have added nine dots . . . and $1 + 3 + 5 + 7 + 9 = 25$. The above series of examples result in a pattern that can be easily shown in a table. Since order in addition does not change the results, reverse the order followed above to make the pattern more obvious:

$$1 = 1$$
$$3 + 1 = 4$$
$$5 + 3 + 1 = 9$$
$$7 + 5 + 3 + 1 = 16$$
$$9 + 7 + 5 + 3 + 1 = 25$$

"How can we extend this pattern?"

$$11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (36)$$
$$13 + 11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (49)$$
$$15 + 13 + 11 + 9 + 7 + 5 + 3 + 1 = \_\_\_\_ \ (64)$$

"How far can the pattern be extended?

r. Factoring and prime numbers. If a sum or a product is given, we can always find at least one pair of addends or factors, respectively, and, sometimes, many pairs of addends or factors. This can lead us to an investigation of how many pairs of whole numbers can be the factors for specific numbers.

Every number can be the product of itself and one, taken as a pair of factors. In every case, except $1 \times 1 = 1$, this particular pair of factors will be two different numbers. "Let's study several examples and find all the pairs of facts there are for each."

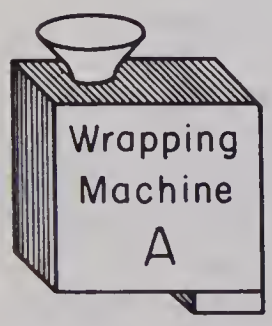
Number taken as a product	Itself and "1"	Other pairs
0	(0,1) (1,0)	(0,7) (0,99) etc.
1	(1,1)	none
2	(2,1) (1,2)	none
3	(3,1) (1,3)	none
4	(4,1)(1,4)	(2,2)
5	(5,1) (1,5)	none
6	(6,1) (1,6)	(2,3) (3,2)
7	(7,1) (1,7)	none
8	(8,1) (1,8)	(2,4) (4,2)
9	(9,1) (1,9)	(3,3)
10	(10,1) (1,10)	(2,5) (5,2)
11	(11,1) (1,11)	none
12	(12,1) (1,12)	(2,6) (6,2) (3,4) (4,3)

Continue this chart as far as time permits, and keep a separate list of all numbers that have two *different* factors, itself and *one*, but for which there are *no other pairs*. (Use of the word *different* is deliberate, because we want to exclude *one* from this list. It will be more useful to us later, if we agree that *one* is not to be considered as a prime number.) The list of numbers having *no other pairs* is:

2, 3, 5, 7, 11, 13, 17, 19, etc.

We have met this sequence of numbers before. We call them *prime numbers* because as products there is only one pair of two different factors that name them: themselves and one. Are all *prime numbers* also odd numbers? (No, there is one exception: 2.)

s. Wrapping machine arithmetic. For this activity, you first make a sketch of a crude machine on the board, labeling it "Wrapping Machine A." Any sketch will do, as long as it has a "place to put things in" and a "delivery station" or output.



	Output	
Input	Number of packages	Contents of each package
6 marbles	2	3 marbles
18 cookies		

"We don't know what's inside this machine. We only know what we put in the machine and what comes out of it. If we put six marbles in through the funnel, the machine gives us two packages, neatly wrapped. When we open the packages, we find that each contains three marbles, what kind of machine is this? Let's try it again. We put in 18 cookies. What do you suppose the machine delivers?" Of course, it might wrap them in packages of three each. Or, it might wrap them in two packages.

"Actually, it gives us two neatly wrapped packages. How many cookies do you suppose we'll find in each package?" Nine, of course.

"Now, let's see if we know what the machine will do in the following situations:

	Output	
Input	Number of packages	Contents of each package
6 marbles	2	3 marbles
18 cookies	2	cookies
30 pennies		
a dozen eggs		
ten trucks and four cars		
a half-dozen apples		
two dimes and six pennies		
three nickels and five pennies		
		7 pencils
		25 stories
		11 letters of the alphabet

The fourth item may cause a feeling of doubt. Some pupils may think the machine couldn't handle such a problem any better than it could handle one marble . . . or one nickel. But most pupils will agree that the machine would give us two packages with six eggs in a package.

With ten trucks and four cars as the problem, the machine might wear a puzzled look—if it had one to wear. The machine might put two cars and five

trucks in each package. That would be consistent. Or, it might put seven vehicles in each package: one containing four trucks and three cars; the other containing six trucks and one car.

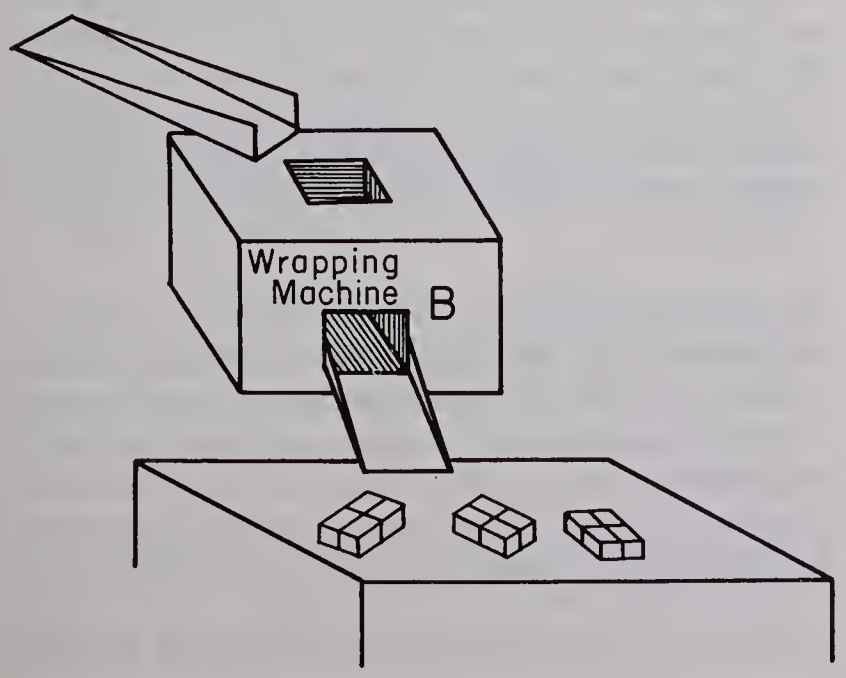
Next item—easy.

Next item—"two dimes and six pennies." Does the machine recognize that a dime is worth 10 pennies? Well, in one package there were four pennies. "What do you suppose was in the other?" Two dimes and two pennies. There were four coins in each package. Evidently, the machine simply counts the objects, and separates them into two groups with the same number in each group.

"Suppose we know the contents of one of the packages. Can we know anything about the 'input'?" If one package contains seven pencils, then the input must have been 14 objects, seven of which were pencils. It is possible that there were 14 pencils, but it is also possible that there were 11 pencils and three erasers, etc.

"What could we feed into the machine that would stump it completely?" Certainly one quarter (25c), one egg, three eggs, etc.

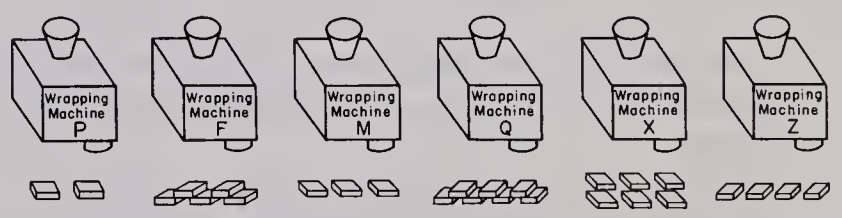
Draw another wrapping machine, "Machine B," that is identical in operation, only it always delivers three packages. If you were to feed it six marbles, you would get them wrapped neatly in three packages, with two marbles in each. Have pupils keep a record of output and input. "How many of the examples of input, that we have considered could both machines handle?" (Six marbles, 30 pennies, a dozen eggs . . . but not 10 trucks and four cars, nor two dimes and six pennies.)



	Output	
Input	Number of Packages	Number items in each package
6 marbles	3	2 marbles
30 pennies	3	10 pennies
1 doz. eggs	3	4 eggs
a set of tricycle wheels	3	1 wheel
1 penny, 4 nickels, 2 dimes, 5 quarters		4 coins
2 doz. eggs		
		7 marbles
		4 nickels, 2 pennies
		3 trucks, 3 cars
		11 stars

This same activity can be extended with machines that deliver four, five, six, etc., packages with the same number of items in each package.

After the idea has been sufficiently developed, draw a series of machines and a series of input-output records on the board. "There has been a mix-up, and the tables (or records) have been separated from the machines they refer to. Can you match each machine with its record?"



Machine M		
Input	No. Pkg.	Items ea. Pkg.
12 coins		
		8 eggs
	3	2 whistles
30 books		
		7 pencils

Machine Q		
Input	No. Pkg.	Items ea. Pkg.
14 marbles		
		10 beads
35 cookies		5 cookies
21 cars		

Machine F		
Input	No. Pkg.	Items ea. Pkg.
10 wheels		
5 apples		
15 cents		
40 stamps		
		4 pears
		$\frac{1}{2}$ doz. eggs
		10 marbles

Machine P		
Input	No. Pkg.	Items
		7 books
		3 rulers
8 pears		
3 doz. bananas		18 bananas

Machine X		
Input	No. Pkg.	Items
1 doz. eggs		a pair of eggs
12 trucks		
		5 books
		10 erasers
		1 apple, 3 oranges

Machine Z		
Input	No. Pkg.	Items
6 prs. shoes		3 shoes
2 doz. Oranges		
		$\frac{1}{2}$ doz. eggs
		7 marbles

Of course, the first record must belong to "Wrapping Machine M"—because the number of packages is shown.

The clue for the second machine is given by the information that it wraps 35 cookies in a certain number of packages—and there are five cookies in each package . . . so, there must be seven packages—and that chart must belong with Machine Q.

Similar clues are given, with varying degrees of directness, for all others except the record that belongs with Machine F. This can be solved in at least two ways: (a) since all other records have been assigned to a machine, the unassigned record and machine must belong together; or (b) assuming the machine can successfully handle all the inputs listed in the record, it must wrap each in five packages because the machines that wrap 2, 3, 4, 6, or 7 packages could not handle all the inputs successfully.

Next, we have enough information now to complete each line of each record. And, we could expand each record as far as we like.

To summarize these results, another chart can be drawn:

	Machine P		Machine M		Machine Z	Machine F	Machine X	Machine Q
Number of Items in Input	In Pkg.	Each	Pkg.		Pkg.	Pkg.	Pkg.	Pkg.
2	2	1						
3			3	1				
4	2	2			4 1			
5						5 1		
6	2	3	3	2			6 1	
7								7 1
8	2	4			4 2			
9			3	3				

Should the chart be extended through inputs of 20 or more, interesting patterns will be detectable. In this chart, if a machine cannot handle a particular "input," the space pertaining to it is to be crossed out. If the class enjoys the "wrapping machine," continue with the idea by asking about this "can't handle" situation. "Machine M wraps every input in three packages. If you fed seven marbles into it, the machine would be stumped. Could a machine be designed to solve this problem some way?" Consider every answer as a possibility. Would the output be confusing? One possible answer would be to have a machine designed so that it would mark the third package, or "only one" or, "There's only one—a full box would be three." This answer anticipates "division remainders," but our main attention is still focused on the idea of multiplication and real life situations where it has application.

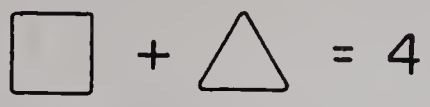
Each episode of this "wrapping machine" story is as open-ended as you want to make it. Each fact of multiplication can be introduced at least once and repeated as often as required. Individual work can be easily assigned. "Make up a chart of your own that might accompany Machine X that has an output of six packages. List 10 inputs and their appropriate outputs. List 10 inputs the machine couldn't handle." The problems can be applied to the other machines.

21. Pairs of answers

A question such as, *How many are 3 and 5?* that can be answered off-hand with a number is usually a trivial question and appropriate only for drill. A question such as, *How many are 357 and 294?* may test the pupil's ability to *regroup* or to *carry,* but the answer again ends with a number: 651.

"Pairs of Answers" is an activity that requires

thinking, but it does not involve large numbers. A notation that may be new for many children is used in asking a question that involves "pairs of answers." For example:



This is an open sentence. It is neither true nor false. The box and triangle are to be considered as *reserved seats* for pairs of whole numbers. If "1" approaches the box or square, and if its partner is "3," they can both sit down because 1+3 = 4. Another pair that would be permitted to take these reserved seats is "0" and "4." In fact, any pair can be seated, provided their sum is 4. A record of pairs with a sum of 4 can be made as follows:

	A	B	C	D	E	F
□	1	0	3			
△	3	4	1			

The pairs mentioned above have been recorded under A and B in the chart. Under C another pair has been recorded. Are there more? Yes, "4" in the square and "0" in the triangle; also, the pair composed of "2" and "2." *That is five pairs. Are there more?* No!

This activity does not involve large numbers, but the questions it raises are not trivial. Pupils are given an opportunity to record the suitable pairs in a pattern that makes an eloquent answer to the question, *Are there any more pairs?*

0 and 4
1 and 3
2 and 2
3 and 1
4 and 0

22. Chain reactions

Given a rule and a pair of numbers, it is possible to set off a *chain reaction* that can be continued across the page, and on indefinitely.

Sample rule: "Each number is to be the sum of the two numbers immediately to the left of it. Given:

1 2 ___ ___ ___ ___ ___ ___ ___ ___

the next term must be 3, because 1 + 2 = 3; the next term is 5, because 2 + 3 = 5; the next is 8, because 3 + 5 = 8; etc. Thus, this chain reaction gives us:

1 2 3 5 8 13 21 34 55 89

and it could keep on going. Change the given numbers, and another chain reaction is under way.

A chain reaction can be worked in both directions. Let the rule remain the same, but change the given numbers and place them mid-point of the reaction:

___ ___ ___ ___ 10 16 ___ ___ ___ ___ ___ ___

Under the rule, the number to the left of 10 must be 6, because 6+10 = 16; the number to the right of 16 must be 26, because 10+16 = 26. So, extended in both directions, the results are:

2 2 4 6 10 16 26 42 68 110 ___ ___

Another sample rule: *Each number must be twice as large as the one to its left.* Now it takes only one given number to start the action.

1	2	4	8	16	32	64
Given	(2×1)	(2×2)	(2×4)	(2×8)		
128	256	512	___	___	___	___	___

Another sample rule: *Each number must be the difference of the two numbers above it: the one on its left subtracted from the one on its right. The bottom row is always the same number.*

1 ___ ___ ___ ___ ___
2 ___ ___ ___ ___ ___
3 ___ ___ ___ ___ ___

Told that the bottom row is all alike, it is possible to begin the reaction:

1 A ___ ___ ___ ___
2 B ___ ___ ___ ___
3 3 3 ___ ___ ___

If 2 in the second line must be gotten by subtracting 1 from A (or A–1 = 2), then A must be 3, because 3–1 = 2. By the same reasoning, B–2 = 3, so B must be 5 because 5–3 = 2. Completed, the chain reaction would look like this:

1 3 8 16 27 41 58 78 101 127
2 5 8 11 14 17 20 23 26
3 3 3 3 3 3 3 3

These chain reactions get underway with a mini-

mum of direction and *go someplace*. They require accuracy, since each in turn depends on its predecessor and any little error along the way will make every succeeding term wrong. Children get a real feeling of satisfaction in being able to demonstrate this accuracy.

23. Fractions

In the material that follows the various expressions used in speaking of fractions will be studied briefly. No rules or procedures should be discussed until pupils have first had an opportunity to acquaint themselves with the whole body of knowledge regarding fractions. Before rules or procedures can be meaningful, this knowledge must be worked into sensible and understandable patterns that are consistent with all the arithmetic of whole numbers pupils have already learned. The approach to fractions used in the *Math Workshop for Children* is to consider this topic not as a separate idea, but rather as an extension of the idea of number, from whole numbers to those numbers which men invented so that there would be an answer to every division problem. Without fractions, $2 \div 3 =$ ____ and $3 \div 2 =$ ____ have no meaning; there is no way to complete these mathematical sentences. We explore fractions to see if they behave like the numbers we know. We examine them to find the answers to these, among other, questions:

1) Can fractions be arranged from small to large?

2) Can pictures be drawn to suggest the idea in the context of a picture that also suggests whole numbers?

3) Are there fractions that are other names for whole numbers?

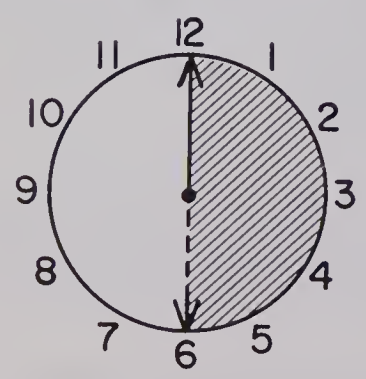
4) Is there at least one fraction to name each whole number?

5) Can fractions help name points on the number line?

6) Can fractions be added? can they be subtracted?

7) Can fractions be multiplied and divided?

Activity 1. Draw a clock on the board, with the minute hand pointing to 12. "Where will the minute hand be in half-hour?" Have a volunteer draw an indication of the hand's second position.



"Color in the area the minute hand covered as it traveled from the first to the second position. How much of the clock face is colored? Have we divided the clock face into two parts about the same size?"

Start over with the minute hand pointing to 12. "Where will the minute hand be in a quarter of an hour? Where will it be a quarter of an hour later? Indicate both positions. Where will it be half an hour from the starting time? Where will it be three-quarters of an hour from the starting time? Indicate this new position. We have shown four of the positions the minute hand passes. Color in the part of the face the minute hand passes over in a quarter of an hour." (If colored chalk is available:) ... "with another colored chalk, shade another part of the face that the hand would pass over in a half-hour."

Follow a similar discussion with one-third and two-thirds of an hour; also with one-sixth, two sixths, three-sixths, four-sixths, five-sixths of an hour.

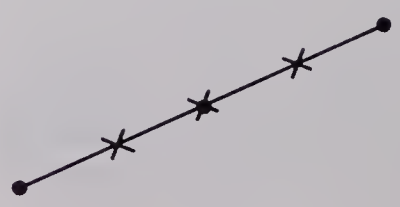
Activity 2. Put a square on the board. "Can someone draw a line that divides the square into 2 parts about the same size? Can someone else draw a different line that divides the square into 2 parts? Any more lines?" (Any straight line drawn through a point where the two diagonals would cross is a correct response. Some child may draw a zig-zag or curved line that would have to be accepted.

Activity 3. Draw a line on the board—preferably at a slight angle—and put a dot on each end.

"This is my picture of a piece of wire. The ends are sharp and I don't want to get scratched, so I've covered them. Where should I cut the wire to make two pieces the same length? Mark the point with a dot.

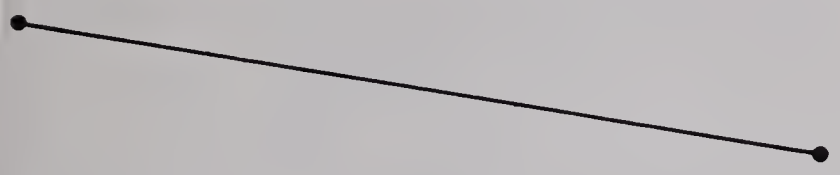
"Suppose I wanted to cut the wire into four pieces that are the same size. Can you mark with a cross the places where I ought to cut it?

"Now I'm going to draw another sketch of my piece of wire. This time I will bend it a little bit.



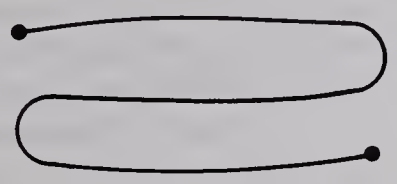
"Where should I cut the wire if I wanted to have two pieces the same size? Can you mark with a dot the place where I should cut it? And where would you put cross marks to indicate where I should cut the wire if I want to have four pieces the same size?

"I found a longer piece of wire."

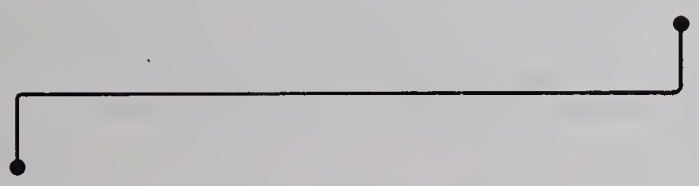


"Where should I cut it, if I want to have two pieces of the same size? Where should I cut it, if I want to have four pieces the same size?

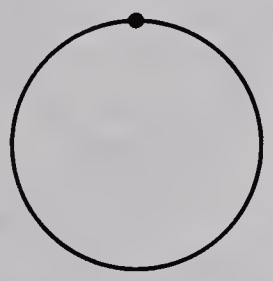
"Now I've bent the wire into an S-shape.



"Now I put two sharp bends in it.



"Finally, I bend it into something like a circle.



"In each instance, mark with a dot the place where I should cut the wire if I want to have two pieces the same size, and then mark with crosses the places where I should cut if I want to have four pieces the same size.

"I finally cut my wire into two pieces.

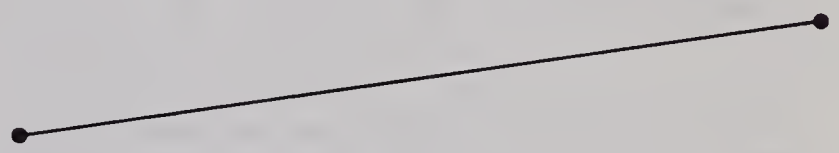
"No sooner had I done this than I was sorry. I had really wanted to cut it into four pieces that were the same size . . . but I only have two. What can I possibly do?" (You will have plenty of advice.)

"If I change my mind again after I have cut the wire in four pieces, is there anything I can do?" (You'll have more advice: solder two pieces, twist them together, glue them, nothing, get another piece of wire, etc.)

We have carefully avoided using the word "equal." This expression is, as previously explained, to be used in referring to the relationship between two names for the same number: $3 + 9 = 12$. Besides, in dealing with wires or sketches, it is not possible to cut them or divide them into "equal" pieces. If we remind pupils that what we really mean when we say "of the same size" is "*about* the same size," there won't be any misunderstanding. "Remember, when we are concerned with answering a question of 'How many,' we must always be exact. But when we are concerned with answering a question that involves any kind of measurement and that asks 'How much,' we can never be exact."

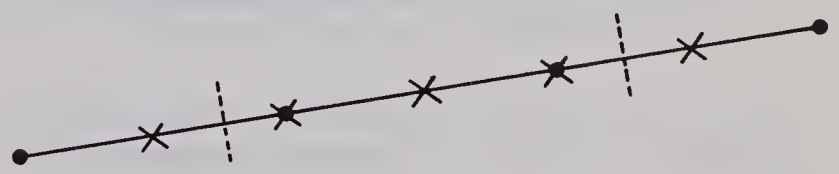
Because the kind of sketches used in this activity are meant only to *suggest* fractions, answers need only be approximate.

Activity 4. "I have a rope. I've knotted each end so it won't unravel. If I were to lay it out straight, it would look something like this:



"If I wanted to cut the rope into three pieces about the same size, where should I cut it? Mark the places with dots.

"Now suppose I wanted to have twice as many pieces of rope, all about the same size. Using a little cross can you mark each place on the rope where I should cut it?



"Why do some of the dots and crosses mark the same place?" (Don't expect well formulated answers. Don't try to settle the matter.)

"If I wanted to have only two pieces about the same size, where would I cut? Is the place marked?

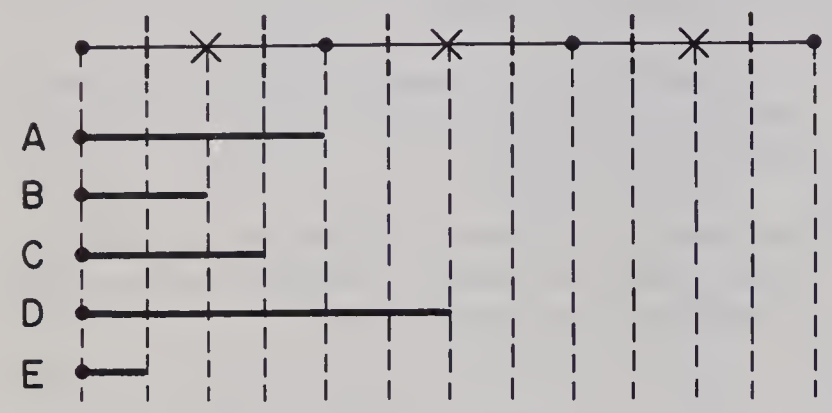
"Suppose I wanted four pieces about the same size? (Cut it in half—and then cut each of the two pieces in half.) Are all the places to be cut marked? Use a dotted line to mark the new places I would need to cut.

"Now, what do you suppose has happened? I've

changed my mind again. I want a dozen pieces – all about the same size. Where shall I cut?" Discuss this matter for a while. You might cut the rope in half and then cut each of those pieces in half to get four pieces. You would then cut each of the four pieces into three parts. Or you might cut the rope into six parts; and cut each of the six pieces in half. Or, you might cut the rope into three parts; and then cut each of those in half, and halve each of the six pieces; etc.

"Are any of the places to be cut already marked? How many more places do we have to mark?" (4) "Indicate these places with dotted lines."

On the board draw the following:



"At one time, I wanted to cut my rope in two pieces about the same size. Does any line on the chart look something like one of those pieces?" (The line marked D.)

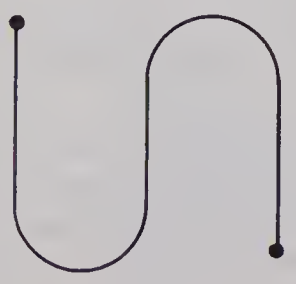
"What do the other sketches suggest to you?

"How many pieces of rope the size of line B would I be able to cut from a piece of rope the size of line D?

"Is the line marked A twice as long as the line marked C? Which line is twice as long as A?"

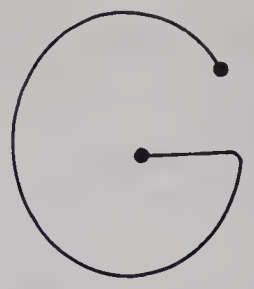
Shift the discussion to a pattern similar to that outlined for page 88 by "bending or twisting your rope."

"Now, I've twisted my rope until it looks like a letter S turned over on its side:



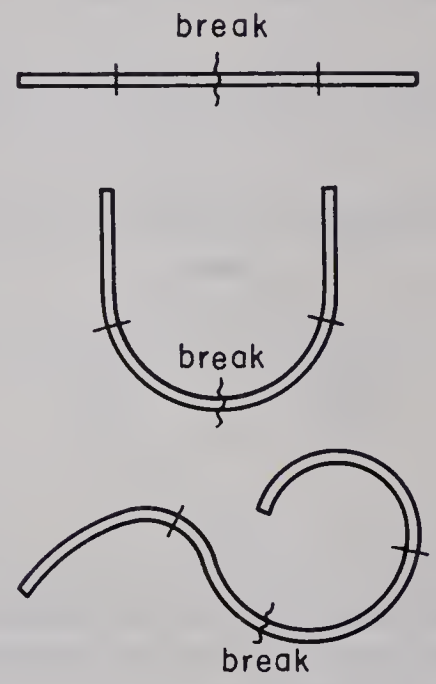
"Use dots to mark the places on the rope that should be cut, if I want to have three pieces about the same size. Next, use little crosses to show where I should cut the rope if I want six pieces about the same size." Do not go further with this example.

"Now, I have twisted the rope into the shape of the letter G."



Have the children mark the places to be cut if you want three and then six pieces.

Activity 5. "Let's talk about some taffy that Martha made. She pulled it into a long piece and layed it on the kitchen table. In what shape do you think she put it down? Well, she might have layed it out straight . . . or in the shape of the letter U . . . or in a variety of other shapes. Here are some possibilities:

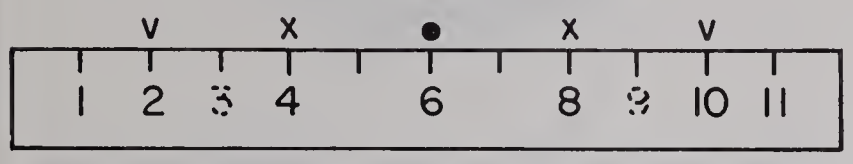


"Martha decided that she wanted to divide the candy into three parts: one for Billy, one for Jean and one for herself. So she marked the places she would break it when the candy got hard." Have pupils make the markings.

"Then Martha remembered that Billy had braces on his teeth and couldn't eat taffy. So, she decided to break it in half when it was hard. She marked the place. "Have a volunteer mark the place." When the candy was hard, Martha broke the large piece into smaller ones. Just then Billy came running into the kitchen. 'Look, Martha, no more braces. Oh, boy, you've made some taffy.' "Now what do

you think Martha did?" She could break each half into three pieces each so that she would have six pieces: two for Billy, two for Jean and two for herself. Or, she could break the taffy at the first marks she made and have two large pieces and two small pieces. Now are the two small pieces together about the same size as one of the large pieces?

Activity 6. Here is a picture of a one-foot ruler.



"Where is the middle of the ruler? What number is usually written at that point on a one-foot ruler? Use small crosses to divide the length of the ruler into three parts about the same size. What numbers would be written at these points?"

"What points would divide the ruler into six parts about the same size? Label the points as they usually are labeled on a ruler."

"Is it easy to divide the ruler into 4 parts about the same size? Label these points if they are not already labeled."

"Are there any more points to be labeled on this ruler?"

Activity 7. "In the shorthand of arithmetic, there are at least three different ways of writing the following sentence." On the board, write: *Six divided by two equals three.*

"How many ways can you write this statement in shorthand, using no written words at all?" The three ways that interest us are given below with 2 examples of each:

$6 \div 2 = 3$ $\quad \frac{6}{2} = 3$ $\quad 2\overline{)6}^{\,3}$

$12 \div 4 = 3$ $\quad \frac{12}{4} = 3$ $\quad 4\overline{)12}^{\,3}$

This suggests that the following statements are true:

$$6 \div 2 = 12 \div 4 \text{ and } \frac{6}{2} = \frac{12}{4} \text{ and } 2\overline{)6} = 4\overline{)12}$$

In each case, read these sentences: "Six divided by 2 equals 12 divided by 4."

"The shorthand $\frac{6}{2} = \frac{12}{4}$ suggests a game. The rules are simple. Make all of the following sentences true, using nothing but this shorthand in which to express division facts. We never use the numeral 1 unless we have to."

$\frac{9}{3} = \frac{12}{?}$ $\quad \frac{16}{4} = \frac{?}{7}$ $\quad \frac{25}{?} = \frac{15}{3}$

$\frac{9}{3} + \frac{4}{2} = \frac{20}{?}$ $\quad \frac{15}{5} + \frac{12}{3} = \frac{?}{2}$

If there is any question about these examples, write them each three ways. For instance:

$9 \div 3 = 12 \div ?$ and $\frac{9}{3} = \frac{12}{?}$ and $3\overline{)9} = ?\overline{)12}$

$(9 \div 3) + (4 \div 2) = 20 \div ?$

$\frac{9}{3} + \frac{4}{2} = \frac{20}{?}$

$3\overline{)9} + 2\overline{)4} = ?\overline{)20}$

After pupils have grasped the idea of the game, move on to examples for which there are several solutions:

$\frac{18}{3} - \frac{18}{2} = \frac{?}{7}$ (because $6 - 4 = 2$);

$\frac{9}{?} + \frac{8}{?} = \frac{?}{2}$ (because $3 + 2 = 5$);

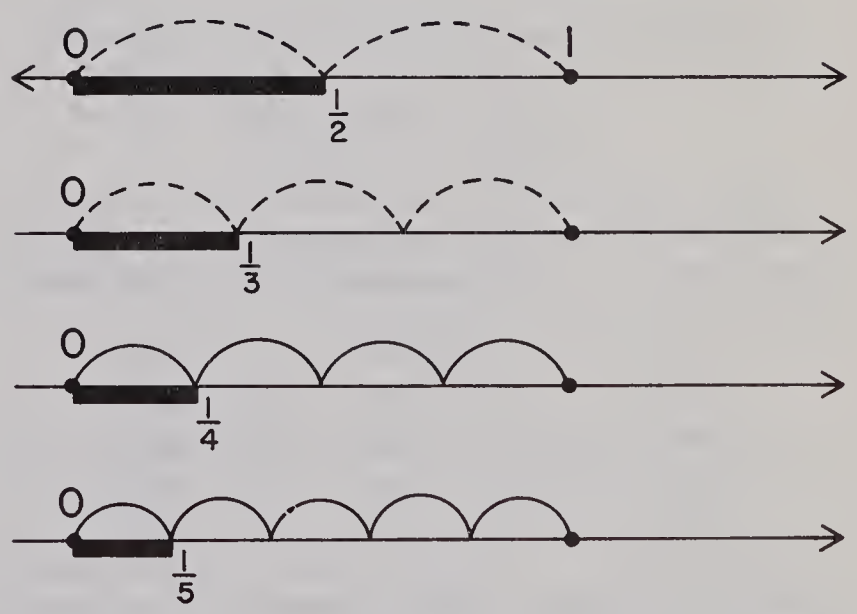
$\frac{?}{2} + \frac{?}{3} = \frac{?}{8}$; $\frac{8}{2} \times \frac{6}{3} = \frac{?}{6}$;

$\frac{20}{2} \div \frac{6}{3} = \frac{?}{5}$; etc.

As pupils play this game, they are on the firm and familiar ground of arithmetic. They will be asked to play the game often. And when they finally come to: $\frac{2}{3} + \frac{3}{4} = ?$ and $\frac{1}{3} \times \frac{2}{5} = ?$, pupils will not find these expressions mysterious. They will have a feeling that such expressions probably fit into the scheme of: $\frac{8}{2} + \frac{6}{3} = \frac{54}{9}$

Activity 8. "Indicate jumps on a number line that suggest;"

$1 \div 2$ or $1/2$	$1 \div 6$ or $1/6$
$1 \div 3$ or $1/3$	$1 \div 7$ or $1/7$
$1 \div 4$ or $1/4$	$1 \div 8$ or $1/8$
$1 \div 5$ or $1/5$	$1 \div 9$ or $1/9$



Have the children complete an example suggesting each of the expressions shown above. When confronted with an example such as $1 \div 4$ or $1/4$, discuss procedures that might be followed, such as: first find the point halfway between 0 and 1; then find the point halfway between 0 and this new point (or,

between 0 and 1/2). When preparing to draw a sketch to suggest the idea of $1 \div 6$ or 1/6, the first step might be the same as in the previous example, or it might be to indicate the points suggesting $1 \div 3$, and then to find the mid-points of each segment. In many cases, the work can begin exactly as for one of the other examples, and then carried further. In such cases, two examples will show their family resemblances. Since 1/6 is related both to 1/2 and 1/3, either of these can be used to begin the work. One-eighth is related in an even closer way to 1/2 and 1/4. One-ninth bears a family relationship to 1/3.

"What labels do not seem to be related to any others in the group under consideration?" (Only 1/5 and 1/7.)

"Do you suppose that 1/5 and 1/7 belong to any families at all?" (Yes, 1/5 is related to 1/10, 1/15, 1/20, etc.; and 1/7 is related to 1/14, 1/21, 1/28, etc.

Refer back to the diagrams suggesting 1/2, 1/3 ... 1/9. "If we want to label all the landing points between 1 and 0, what labels would you suggest in each instance?" (2/3, 1/3; 3/4, 2/4, 1/4; 4/5, 3/5, 2/5, 1/5; etc.)

"Starting from 0, which is larger, a jump to 3/5 or a jump to 1/2?"

"In the following pairs, circle the member that is located further away from 0 on the number line."

1/3, 2/3	3/4, 1/4
2/4, 1/5	2/3, 5/6
2/8, 1/5	1/2, 3/7

"Consider the following. Which is further from 0?"

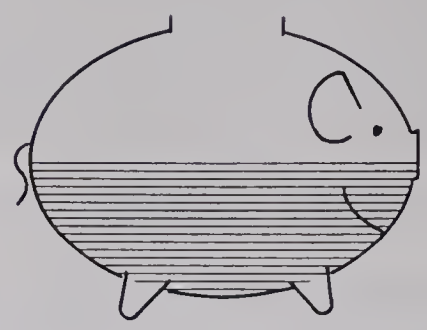
1/2, 3/6	2/4, 1/2
2/6, 1/3	2/3, 6/9

"We have a shorthand that expresses these ideas. I will write some true statements in this shorthand. Can you complete the others?"

$1/2 > 1/3$	$1/2 < 3/4$	$1/2 = 3/6$
$3/9 = 1/3$	$3/6 < 4/6$	$7/8 > 1/4$
2/4 3/6	2/3 2/4	1/7 1/4
1/3 2/7	3/5 1/2	4/8 3/6

Continue examples until everyone can use this notation. Then explain that ">" is shorthand for "is greater than . . ." and "<" is shorthand for "is less than . . ." Point out that if the shorthand symbols reminds one of an arrow, the thin or sharp-end points to the "smaller of the two," and the wide part of the symbol is close to the "larger of the two."

Activity 9. "Here is a sketch of a piggy bank. It is half-full of pennies."

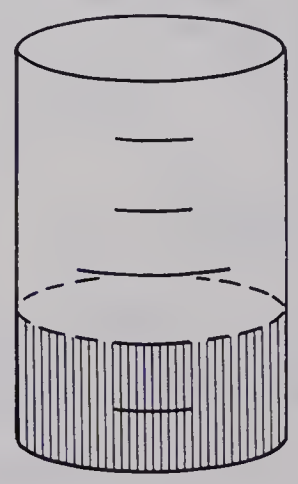


"Can you tell me how many pennies there are? What do you need to know before you can tell me the approximate number? If I told you how many pennies were needed to fill the piggy bank, could you tell me about how many pennies the bank holds when it is half-full?"

Value when Full	No. of Pennies when half-full
$1.00	
$ .50	
$3.00	

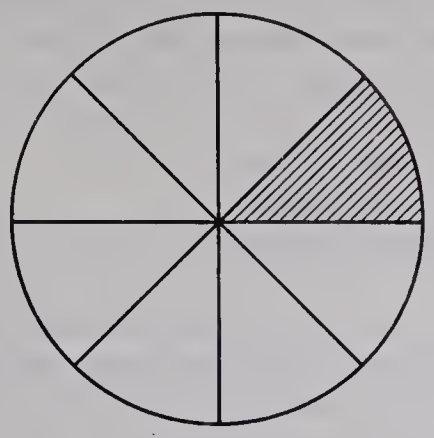
Much of mathematics can be stated in the form of "*If..., then....*" propositions. *If* a full piggy bank holds 150 pennies, *then* about how many pennies does it hold when it is half-full?

Activity 10. "Here is a water tank. The marks on it help to indicate how full it is."



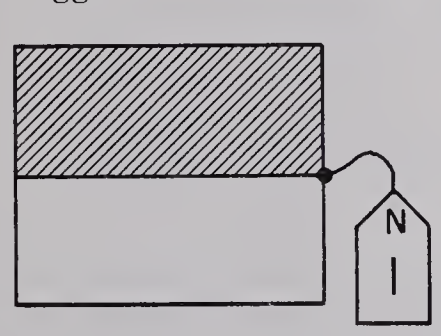
No. of Gallons when full	No. Gallons in shaded area
6	
60	
3	
1	

Activity 11. "Here is a pie that is cut into 8 pieces. There are 16 ounces in a pound. How many ounces would there be in a shaded piece?"



Weight of Whole pie	Weight of 1 piece	
In pounds	In pounds	In ounces
1	1/8	2
2		
4		
6		

Activity 12. "Here is a sketch of 1 of something. This is indicated by the 1 on the tag. What does the shaded area suggest? The unshaded area?"



Shaded	Unshaded	Tag
1/2	1/2	1
		2
		1/2

"Suppose we change the label or tag. Let's write 2 in the tag. Now, what do the shaded and unshaded areas suggest?" (1 and 1)

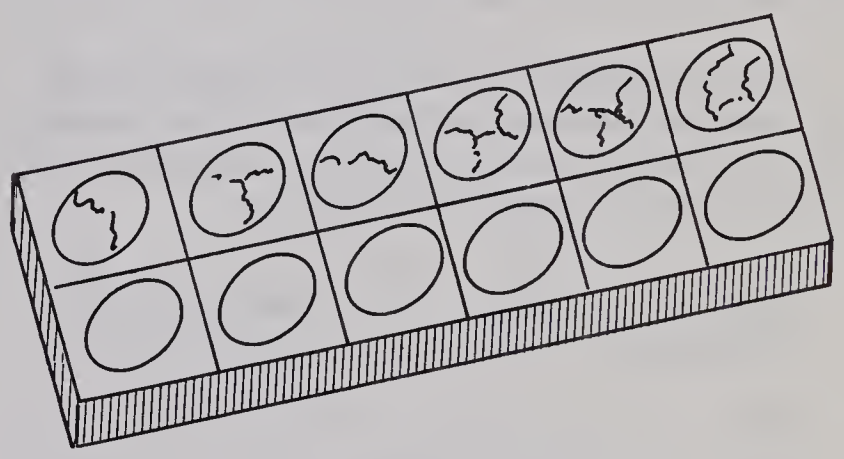
"Suppose we change the tag to read 1/2. Now, what does the shaded area suggest? (1/4 and 1/4) Without discussing, make the comment—"It seems that a half of one-half is one-fourth. In money this would mean that half of a half-dollar is the same as a quarter of a dollar."

"Can you make sketches suggested by each row in the following chart?"

Shaded	Unshaded	Tag
3/4	1/4	1
1/3		1
	2/5	1
3/2		2
	5/3	2

Don't hurry. Discuss the adequacy of each sketch, possible improvements, similar sketches that suggest the same idea, similar ideas suggested by the same sketch, etc.

Activity 13. "Bill said that 6 eggs in a carton of 12 were cracked. His sister said, 'But it looks to me as if 3 in each half-dozen are broken.' His little brother said, 'In each group of two eggs, one is broken.' When his Mother looked at the carton she said, 'Why half of them are broken.' "



"Here is a sketch of the situation. Who was right?" Write on the board:

Bill: "6 eggs are cracked."
Sister: "3 in each half-dozen are broken."
Little brother: "One out of every 2 is broken."
Mother: "Half are broken."

"Are they all saying much the same thing? In mathematics, we say that numbers have many names. We say, for example, that:

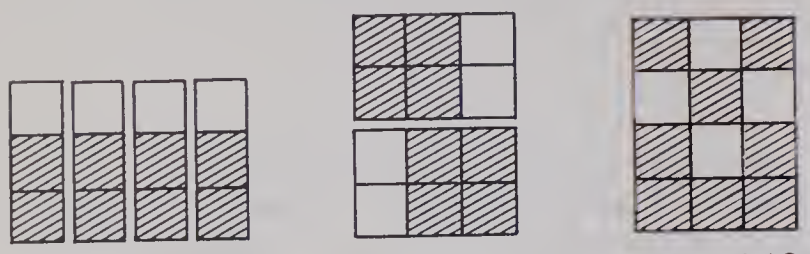
$$\underline{1/2} = 2/4 = \underline{3/6} = 4/8 = 5/10 = \underline{6/12}$$

"The underlined names for 1/2 suggest something quite similar to the remarks about the broken eggs."

Activity 14. Cut squares about 2 × 2" out of cardboard that is white on one side, gray on the other. Or have the quares colored one color on one side, another color on the other. Put 12 squares in a box. Mix them up and dump them out on a table.

"How many squares are white? How many are gray? Without turning any over, arrange them in

a 3×4 array. Arrange them in different patterns." (Unless the result is 5 of one color and 7 of another, they can usually be arranged and rearranged to indicate many ways of saying the same thing. Suppose there are 4 white and 8 gray:



grey – 2 out of 3
white – 1 out of 3

4 out of 6
2 out of 6

8 out of 12
4 out of 12

After several examples, add 4 more squares so that there are 16 in all. Now all even numbers of squares, either white or gray (if one is even, so must the other be even), can be arranged in more than 1 way. The most useful results for making our point here are 8 – 8 and 12 – 4.

"The shorthand we have been using: 1/3, 2/5, 4/8, etc., can be used to keep track of our results. In the example when 8 out of 12 were gray, we can express the idea in the sketch this way:

Gray	White	All together
8/12 or 4/6 or 2/3	4/12 or 2/6 or 1/3	12/12, 6/6, 3/3, 11/11, 10/10, 9/9, etc.

"We can read this new shorthand: '8 out of 12 are gray' or '4 out of 6 are gray,' etc." Write the following on the board, and have pupils complete them.

If 8/12 are gray, then are white?
If 3/12 are white, then are gray?
If are white, then 7/12 are gray?

"Are the following statements true?"

If 8/12 are gray, then 1/3 is white?
If 1/4 is white, then 9/12 are gray?
If 2/12 are white, then 5/6 are gray?

Add 8 more squares. Now there are 24 in all. Proceed in same general way.

Talk about each example. Draw sketches. Encourage the children to explain their arguments. Keep asking, "Does that sound reasonable? Does everyone agree?"

Do not expect to reach any general rules about this idea that, later on, will be referred to as "ratio." We are introducing the idea at this point mainly for discussion purposes. We will come back to it.

Use the 2 × 2" squares of white-gray cardboard (or equivalent) for an early experience with the idea of chance or probability.

Put two squares in a box. Mix them, and drop them on a table. "How many are gray? In what different ways can squares show up?"

0 out of 2 may be gray
1 out of 2 may be gray
2 out of 2 may be gray

"Are there any other possible outcomes? Which result do you think will happen most often?"

Let several groups experiment, and have them each keep a record:

	How many are gray?		
	0/2	1/2	2/2
Totals	6	14	4

The more trials each group makes, the more the results will approximate a 1-2-1 ratio. That's a secret.

Then make a compilation of totals.

Group	0/2	1/2	2/2
A	6	14	4
B	4	10	8
C	7	15	4
D	8	13	8
Totals	25	52	24

Expand the activity to investigation of results when three squares are used. "What are the possible combinations? How many can be gray each time?"

0 out of 3
1 out of 3
2 out of 3
3 out of 3

Again, keep records.

	How many are gray?			
	0/3	1/3	2/3	3/3
Totals	5	14	16	6

Again, have several groups keep track of as many trials as possible—and then record the totals as

shown above. (This time, the outcomes will approximate 1-3-3-1—*that's a secret.*)

The only goals we have at this point are to plant the idea that certain combinations seem to occur more often than others; and to use familiar notations to express ratios. (For example, 2/3, meaning "2 out of 3.") However, we do not yet refer to the term *ratio*.

Activity 15. Experiment with notations and sketches such as follows:

"Here are two arrays of 15 squares. On one, we will mark one out of every three squares with an A. On the other, we will mark two out of every five squares with a B. How many squares out of the 15 will be marked in each case?"

A	A	A	A	A

1 out of 3,
or 5 out of 15

B	B			
B	B			
B	B			

2 out of every 5,
or 6 out of 15

Next show the same number of squares marked A and B together on a single sketch of 15 squares:

A	A	A	A	A
B	B			
B	B		B	B

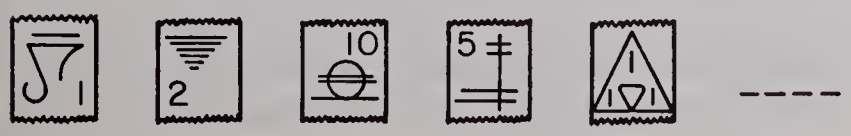
"If we were to simply put one diagram on top of the other, two squares would contain 2 marks. Since we want only one mark in any one square, we must shift two of the B's.

5 out of 15 and *6 out of 15* together are *11 out of 15*; or *1 out of 3* and *2 out of 5* together are *11 out of 15.*

"In shorthand, we might write:

5/15 and 6/15 → 11/15, or
1/3 and 2/5 → 11/15

Activity 16.
"Alice saved stamps. Her friend, Maxine, also saved stamps. Alice decided to give Maxine one out of each pair of duplicates. So, she arranged them in pairs across ther desk, something like this:



"How many did she give Maxine? Do you have enough information? Which of the following sentences need to be completed?

A. Alice has __ stamps arranged in pairs on her desk.
B. The stamps were worth $__.
C. She had __ pairs arranged on her desk.
D. Alice gave Maxine __ out of every pair of stamps.
E. Alice had __ stamps of which she had no duplicates.
F. Of those so arranged, she kept __ stamps.

Completions for any one of the sentences A, C, D, or F will provide enough information to answer the question conclusively.

Without completing A, C or F, have pupils draw a chart of some of the possibilities, if they know how many stamps Alice gave Maxine:

Number to Maxine	1	5	50		
No. of stamps arranged in pairs	2	10	100	etc.	

"In grocery stores, you will find cans, cartons, bottles and packages marked this way:

3/15	2/11	4/25
5/20	5/22	5/13

"What do you think those marks mean?" (3 for 15¢; 2 for 11¢; 4 for 25¢; etc.)

Activity 17. "If a can is marked 3/15, how much would you pay for 1 can?" Someone will say that if 3 cans cost 15¢, then 1 can would cost 5¢ because 15 ÷ 3 = 5. Someone else might contend that such is not the case, because then the cans would be marked 1/5. If the price is meant to indicate a bargain for purchasing 3 cans, then the price for 1 can must be 6¢. By purchasing 3 cans for 15¢ the customer saves 1¢ per can.

On the board draw a chart for each case:

No. of Cans	1	2	3	4	5	6	7	8	9	10
Price in cents	*5*	*10*	15			30				

No. of Cans	1	2	3	4	5	6	7	8	9	10
Price in cents	*6*	*12*	15	*21*	*27*	30				

"We agree that the price is 3 cans for 15c in either case. How much would 6 cans cost? How much would 9 cans cost? The same in both records? Now, how about 1 can? We have agreed that I can might cost 5c or 6c. In each case, how much would 2 cans cost?" (10c in the first case; 12c in the second.) But someone might argue that if 1 can costs 6c and 3 cans cost 15c, then you probably get a small bargain for buying 2 cans at a time. So, while 1 can cost 6c, 2 cans may cost only 11c. That would look like this:

		Savings
1 can	6c	0
2 cans	11c	1c
3 cans	15c	3c

So, we need another chart:

No. of Cans	1	2	3	4	6	7
Price in cents	*6*	*11*	15	5	30	

A final word

The *Math Workshop for Children* has not invented a new mathematics! Nor have its authors spun novel theories of education. Rather, in it they have tried to bring into a whole the most universally accepted ideas of both educators and mathematicians.

Basically, then, the *Math Workshop for Children* takes hold of traditional subject matter and prescribes for its study large doses of solid mathematics, rather than the traditional presentation of isolated facts.

We have accepted the challenge of mathematicians and educators to think about the problem long enough and with the creative interest it takes to distill mathematical ideas until their essence emerges in the simplicity that new young minds insist upon—and respond to.

Robert W. Wirtz
Morton Botel
W. W. Sawyer

Notes to accompany *Work Text* pages 1-2

Everyone is sure he knows that Alec's machine is simply doubling the number written on the screen. If you assume that, your predictions will all come true. But, are you sure that that is the procedure the machine follows?

Judge Harris isn't sure. He says: "Add 5, double the result, and subtract 10."

Judge Thornton has another theory. He says: "Multiply by 6 and divide by 3."

There are, of course, many theories that might explain the procedure the machine might be following.

Urge the children to make up their own theories about ways the machine might use to get the same results. Some may be very obvious, such as: "Double the number, add 2 and then subtract 2," or "Double, subtract 3 and add 3," etc.

Play a game based on this same idea. Two teams can play—team A and team B. First, team A writes a number. Team B selects a rule and, applying the rule, answers with the result.

As soon as team A is sure it knows "the rule," it reports: "We are going to write ____ (whatever they decide) and you will write ____."

If team A is correct, that round is completed, and team B gets one point for every number A wrote down. If team A is wrong, team B gives the correct answer and has five points added to their score.

For example. Suppose team B decides on the rule: "Double and subtract 4." Here is a score board:

	1	2	3	4	5	
A writes	4	5	7			
B writes	4	6	10			

A calls out "4," and B answers "4"—because $2 \times 4 = 8$ and $8 - 4 = 4$.

A calls out "5," and B answers "6"—because $2 \times 5 = 10$ and $10 - 4 = 6$.

Now team A has a decision to make. It may see its numbers as beginning a sequence of 4, 5, 6, 7 and the numbers B uses as answers as a sequence of 4, 6, 8, 10. So, if they call out 6, B will answer 8. Are they sure? If they are wrong, B will get five points and the round will continue. Guessing can be expensive.

They decide to risk another point rather than guess. They call out "7," and B answers "10."

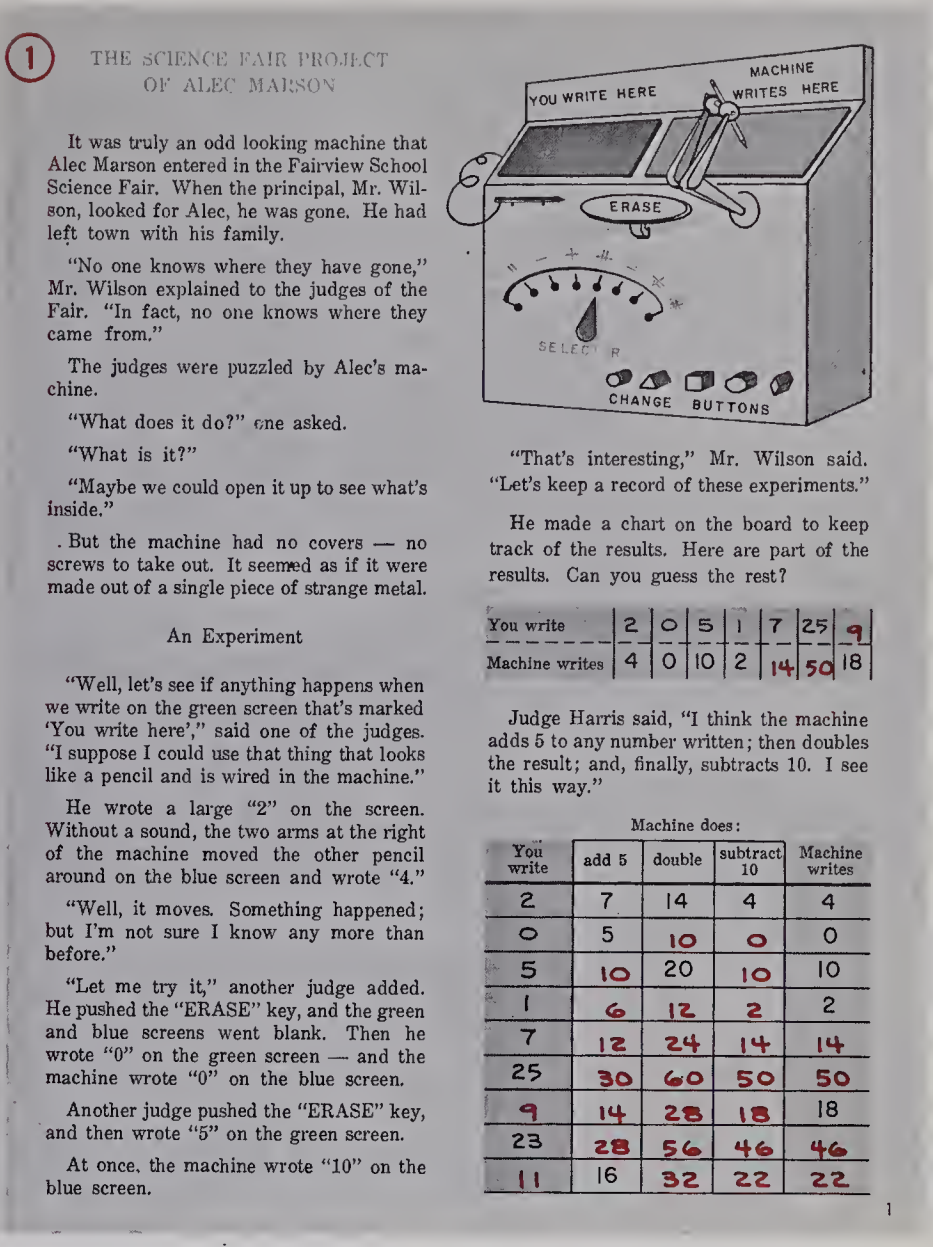
Now they feel quite sure. They look at it this way:

	4	5	6	7
	4	6		10

"If we call 6, you will answer 8."

"That's right," B answers. That's the end of the round, and B has collected 3 points—because B had to give three answers before its rule was discovered.

Now A makes up the rule, and B calls out the numbers. A decides on the rule: "Subtract 2 and triple the result." B calls out "5," and A answers with "9"—because $5 - 2 = 3$ and $3 \times 3 = 9$. B calls out "1," and A is in trouble unless the children are familiar with "negative and positive" or "directed" numbers and know what 3×1 means. If not, then



1 THE SCIENCE FAIR PROJECT OF ALEC MARSON

It was truly an odd looking machine that Alec Marson entered in the Fairview School Science Fair. When the principal, Mr. Wilson, looked for Alec, he was gone. He had left town with his family.

"No one knows where they have gone," Mr. Wilson explained to the judges of the Fair. "In fact, no one knows where they came from."

The judges were puzzled by Alec's machine.

"What does it do?" one asked.

"What is it?"

"Maybe we could open it up to see what's inside."

But the machine had no covers — no screws to take out. It seemed as if it were made out of a single piece of strange metal.

An Experiment

"Well, let's see if anything happens when we write on the green screen that's marked 'You write here'," said one of the judges. "I suppose I could use that thing that looks like a pencil and is wired in the machine."

He wrote a large "2" on the screen. Without a sound, the two arms at the right of the machine moved the other pencil around on the blue screen and wrote "4."

"Well, it moves. Something happened; but I'm not sure I know any more than before."

"Let me try it," another judge added. He pushed the "ERASE" key, and the green and blue screens went blank. Then he wrote "0" on the green screen — and the machine wrote "0" on the blue screen.

Another judge pushed the "ERASE" key, and then wrote "5" on the green screen.

At once, the machine wrote "10" on the blue screen.

YOU WRITE HERE
MACHINE WRITES HERE
ERASE
SELECTOR
CHANGE BUTTONS

"That's interesting," Mr. Wilson said. "Let's keep a record of these experiments."

He made a chart on the board to keep track of the results. Here are part of the results. Can you guess the rest?

You write	2	0	5	1	7	25	9
Machine writes	4	0	10	2	14	50	18

Judge Harris said, "I think the machine adds 5 to any number written; then doubles the result; and, finally, subtracts 10. I see it this way."

Machine does:

You write	add 5	double	subtract 10	Machine writes
2	7	14	4	4
0	5	10	0	0
5	10	20	10	10
1	6	12	2	2
7	12	24	14	14
25	30	60	50	50
9	14	28	18	18
23	28	56	46	46
11	16	32	22	22

1

2

ANOTHER SUGGESTION

Judge Thornton said, "Perhaps the machine multiplies the number you have written by 6, and then divides the result by 3. If it did, the results would be:"

You write	Machine's Work: multiply by 6	Machine's Work: divide by 3	Machine writes
2	12	4	4
0	0	0	0
5	30	10	10
1	6	2	2
7	42	14	14
25	150	50	50

Mr. Wilson, the principal, suggested that perhaps the machine wasn't as complicated as the judges thought.

"Maybe it simply reads what you write and then doubles that number."

But Judge Jones was ready with another explanation.

"It could be that the machine adds 3, triples that sum, subtracts 8 from the result, and then takes away 1 more than the number you write."

You write	add 3	triple	subtract 8	subtract 1 more than no.	Mach. writes
2	5	15	7	4	4
0	3	9	1	0	0
5	8	24	16	10	10
1	4	12	4	2	2
7	10	30	22	14	14
25	28	84	76	50	50
9	12	36	28	18	18
23	26	78	70	46	46
11	14	42	34	22	22
10	13	39	31	20	20
17	20	60	52	34	34
50	53	159	151	100	100

MORE EXPLANATIONS

Other suggestions were as follows:

a. "Multiply by 10 and divide by 5."

b. "Multiply by 3 and subtract the number you wrote."

c. "Halve the number you wrote and multiply the result by 4."

d. "Subtract 5; double the result, and then add 10." (Do you think the machine can subtract 5 from 0 or from 1 or 2 or 3 or 47)

Mr. Wilson was a little impatient. "I'm sure," he said, "that Alec Marson could have built a machine that got the same results by an even more complicated procedure.

"He might have had his machine add 2 to the number you write; multiply by 6, add 9, divide by 3 and subtract 7. Here are some results:"

You write	2	0	5	1	7	3	4
add 2	4	2	7	3	9	5	6
multiply by 6	24	12	42	18	54	30	36
add 9	33	21	51	27	63	39	45
divide by 3	11	7	17	9	21	13	15
subtract 7	4	0	10	2	14	6	8
Machine writes	4	0	10	2	14	6	8

"However," Mr. Wilson continued, "Alec Marson was a very bright boy. He seldom used complicated means when the same thing could be done more easily.

"I'm convinced he built a machine that doubles any number you write on the green screen.

"I think that there is a lot more to Alec's machine than we know about yet. Let's see what else it will do."

A simply replies: "Your number is too small; pick another" . . . and the score remains unchanged.

B calls out "2," and A answers "0"—because $2 - 2 = 0$ and $3 \times 0 = 0$. Now the score board looks like this:

	1	2	3	4		
B	5	2				
A	9	0				

That's not much help. So B works out a plan of action. They will fill in one more column in a chart and see if it reveals a pattern:

2 → 0
3 → 3
4 → ___
5 → 9

So they decide to call out "3," and A answers "3" . . . which B sees as another item in its chart. Now B has to decide what A will answer if it calls out "4." The pattern seems to suggest risking "6" to fill out the 0, 3 ___, 9 pattern.

"If we call 4, you will answer 6."

"That's right."

So B gets three points. And another round begins.

Suppose B thinks up a tricky rule: "Divide by 2 and answer with the remainder."

A calls out "5," and B answers "7." A calls out "6," and B answers "0." So the scoreboard looks like this:

	1	2	3	4	
A	5	6	8	9	
B	1	0	0	1	

A decides to make a guess. It seems to them that B is simply subtracting 6, so they decide to say: "If we call out 8, you will answer 2."

"You're wrong. If you call out 8, we will answer 0." So that is recorded and B gets five points added to its score.

A looks carefully now at the situation. With what kind of rules does 6 lead to 0 and 8 lead to 0? Maybe all even numbers go to 0 and odd numbers go to 1. Some members of team A question that B would think up such a tricky rule. Maybe the rule is: "All numbers 5 or less go to 1, and all numbers more than 5 go to 0."

Now, if team A calls out 3 and B answers 1, they will still not know the rule. So, they decide to call out 9; that is more than 5 and would go to 0 under one rule, but it is odd and would go to 1 under the other rule. So, they chose that course of action. It will cost them one point, but it would have given five more points to B if they made another mistake.

They call out "9."

The answer from B is "1."

"We know now," A says with confidence. "If we call out 4, you will say 0."

"That's right." And B gets four more points because it gave four answers before A could make a correct prediction.

Play this kind of game often. Children will spend spare time thinking up tricky rules—the trickier the better. It will take considerable teamwork to solve

some of the rules. Their attacks on hard examples will draw them deep into mathematics.

This game will be an appropriate one to play very often as *Work Text Level E* unfolds involving common fractions, decimal fractions, divisibility, squares of numbers, etc. As children acquire greater command of such ideas, their rules reflect an increase in mathematical sophistication.

Notes to accompany *Work Text* page 3

On page 3, the Cross Number Puzzle is introduced for the first time in *Level E*. Along with its cousin, "Magic Squares," this fascinating device will be employed often in the next 10 pages.

Mr. Wilson explains the rules for the simplest Cross Number Puzzle: Add the numbers noted in the top row, and write the sum at the right. Do the same with the second row. Then, add the numbers noted in the left-hand column, and write the sum at the bottom. Do the same with the other column.

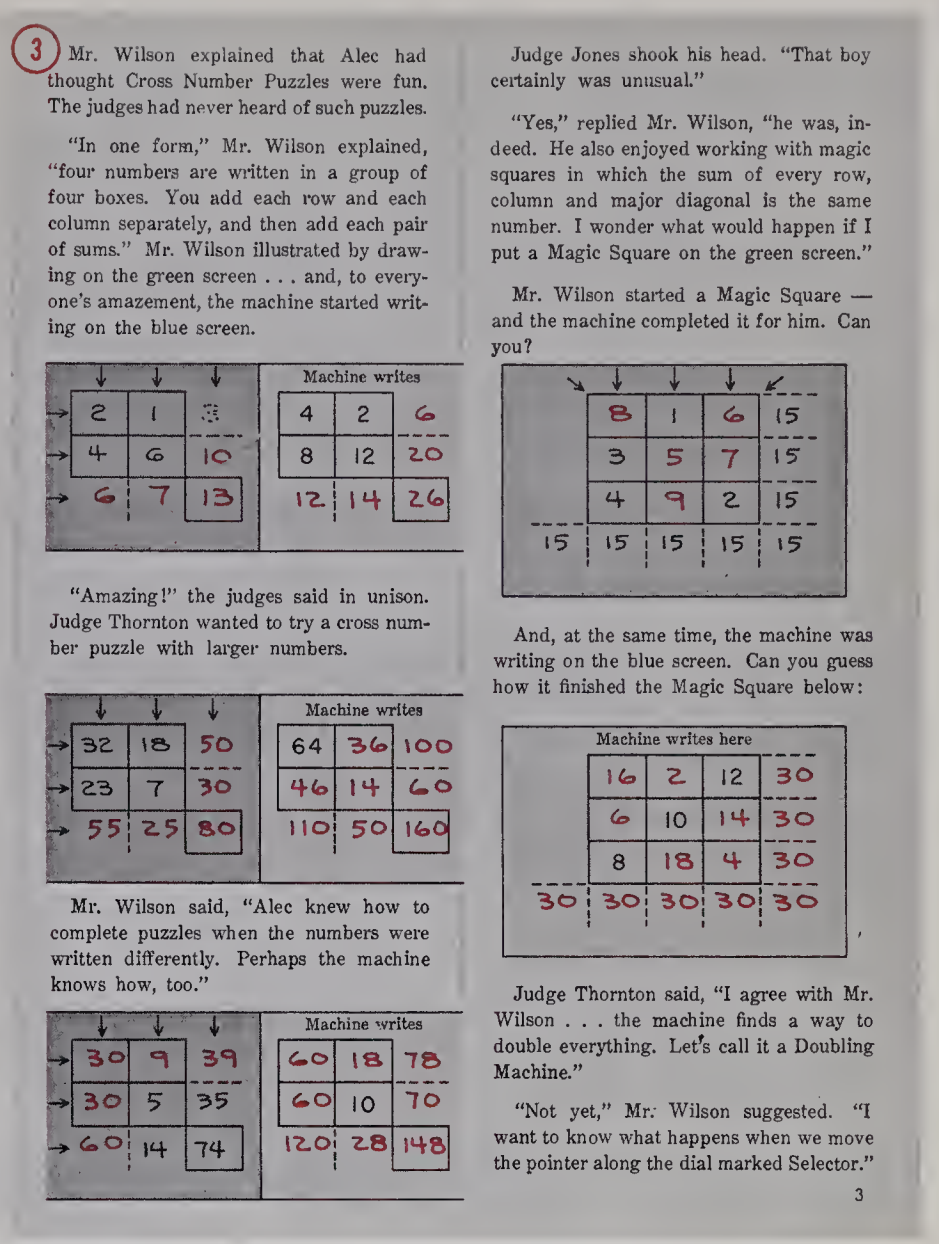
Now, if the puzzle "works," you can write a number in the check box (lower right-hand corner, in this case) that is the sum of the two numbers noted above it and the two on the left, as in the first example: $13 = 10 + 3$ and $13 = 6 + 7$. Refer to the *Preface* for a discussion of the rich mathematics that underlies this disarming little puzzle.)

But as soon as Mr. Wilson put such a puzzle on one screen on the machine, the machine put another on the other screen. What is the relation between the two puzzles . . . or is there no relation?

Instead of trying to use words to explain the pattern they see, the judges try another example—and test their ability to predict results.

In the third pair of puzzles at the bottom of the left column, the puzzle requires some reverse arguments. If 75 is written in the check box and 35 is the sum of the numbers in the bottom row, then the sum of the top row must have been 39, because $74 - 35 = 39$ and $35 + 39 = 74$. Since $74 - 14 = 60$ and $14 + 60 = 74$, then 60 must be the sum of the numbers in the left column.

Further, if 35 is the sum of the bottom row and 5 is one of the addends, then the other must be 30.


3
Mr. Wilson explained that Alec had thought Cross Number Puzzles were fun. The judges had never heard of such puzzles.
"In one form," Mr. Wilson explained, "four numbers are written in a group of four boxes. You add each row and each column separately, and then add each pair of sums." Mr. Wilson illustrated by drawing on the green screen . . . and, to everyone's amazement, the machine started writing on the blue screen.
Machine writes
2 1 3
4 6 10
6 7 13
4 2 6
8 12 20
12 14 26
"Amazing!" the judges said in unison. Judge Thornton wanted to try a cross number puzzle with larger numbers.
Machine writes
32 18 50
23 7 30
55 25 80
64 36 100
46 14 60
110 50 160
Mr. Wilson said, "Alec knew how to complete puzzles when the numbers were written differently. Perhaps the machine knows how, too."
Machine writes
30 9 39
30 5 35
60 14 74
60 18 78
60 10 70
120 28 148
Judge Jones shook his head. "That boy certainly was unusual."
"Yes," replied Mr. Wilson, "he was, indeed. He also enjoyed working with magic squares in which the sum of every row, column and major diagonal is the same number. I wonder what would happen if I put a Magic Square on the green screen."
Mr. Wilson started a Magic Square — and the machine completed it for him. Can you?
8 1 6 15
3 5 7 15
4 9 2 15
15 15 15 15 15
And, at the same time, the machine was writing on the blue screen. Can you guess how it finished the Magic Square below:
Machine writes here
16 2 12 30
6 10 14 30
8 18 4 30
30 30 30 30 30
Judge Thornton said, "I agree with Mr. Wilson . . . the machine finds a way to double everything. Let's call it a Doubling Machine."
"Not yet," Mr. Wilson suggested. "I want to know what happens when we move the pointer along the dial marked Selector."
3

With such arguments, the puzzle can be completely reconstructed. Then the relationship between the puzzle of the judges and the puzzle with which the machine responds, enables us to predict the machine's next move.

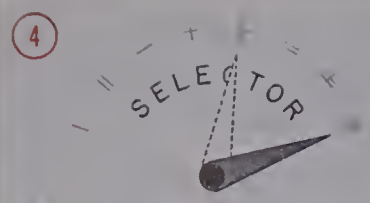
When Mr. Wilson tries a magic square, the machine adopts the same response pattern.

"Is the response of the machine also a magic square?" (Yes.)

Notes to accompany *Work Text* pages 4-5

Pages 4 and 5 begin to reveal the great flexibility of the machine Alec Marson entered in the School Science Fair. So far, it seems to be doubling everything, but no one has moved the selector. Mr. Wilson decides to move it. Now what is the machine going to do? There's only way to find out: experiment. Write on one screen—and study the response the machine makes.

If you write 4 on the screen and the machine writes 16, there are so many rules the machine might be following that would lead to different results. Here are some alternatives we could expect:



Mr. Wilson turned the pointer from # to #. Then he wrote 4 on the green screen — and the machine wrote 16 on the blue screen.

"Ah!" exclaimed Judge Thornton, "now it's multiplying by 4. This is truly a wonderful machine."

After pushing the Erase key, Mr. Wilson wrote 6 and the machine wrote 18.

"What's that? Did the machine make a mistake?"

Mr. Wilson made a record of his results. Can you complete the blanks?

You write	4	6	12	9	13	24	45
Machine writes	16	18	24	21	25	36	57

"No," Mr. Wilson concluded, "the machine did not make a mistake."

"I think I see what's happening," Mr. Thornton said. "The machine adds 6 to the number written, doubles the result and subtracts the number you wrote. I would show it this way:"

You write	4	6	12	9	13	24	45
add 6	10	12	18	15	19	30	51
double	20	24	36	30	38	60	102
subtract no. you wrote	16	18	24	21	25	36	57
Machine writes	16	18	24	21	25	36	57

Judge Jones had been scratching his head. He was puzzled by their results. And then his face lit up.

"I've got it!" he said. "The machine doubles the number written, adds 24 to the result, and then divides by 2. Like this:"

You write	4	6	12	9	13	24	25
double	8	12	24	18	26	48	50
add 24	32	36	48	42	50	72	74
divide by 2	16	18	24	21	25	36	37
Machine writes	16	18	24	21	25	36	37

By this time, Judge Harris had his own idea of the way the machine worked.

"Add 4," he said. "Triple the result; subtract twice the number you wrote. Like this:"

You write	4	6	12	9	13	24	25
add 4	8	10	16	13	17	28	29
triple	24	30	48	39	51	84	87
subtract twice no. you wrote	24 − 8	30 −12	48 −24	39 −18	51 −26	84 −48	87 −50
Machine writes	16	18	24	21	25	36	37

Mr. Wilson was growing impatient again.

"No," he said. "Alec always did things the easy way. He avoided complications. I'm sure that when the selector points to #, the machine reads the number you wrote, adds 12 to it and writes the result."

"Yes, of course," Judge Harris admitted. "We certainly missed the easy way. Let's try another Cross Number Puzzle."

You write here			Machine writes here		
9	7	16	12	10	22
20	30	50	23	33	56
29	37	66	35	43	78

↓

you write	4	0	5	2	7
machine writes	16	8	18		

(Rule might be add 4 and double result)

Have children find rules of their own that start out with the result that 4 leads to 16, and yet yield a *new* set of results.

We say "new" because we want to put the spotlight on any rules that repeat earlier results. For example, the rule on *Work Test* page 4 might be "add 12." Suppose someone suggests a rule of "add 20 and subtract 8 from the result." Why would this produce the same result as found on page 4? Why does "add 12" and "add 20 and subtract 8" act as a single rule? Could it be because $12 = 20 - 8$?

The study of such almost obvious equivalent rules will help children understand why some of the complicated rules the judges dream up produce the same results.

↓

you write	4	2	5	9	12	
machine writes	16	8	20			

(Rule might be multiply by 4)

↓

you write	4	2	1	5	3	6
machine writes	16	4	1	25		

(Rule might be multiply number by itself)

↓

you write	4	2	5	8	3	6
machine writes	16	0	24	48		

(Rule might be subtract 2 and multiply by 8)

Lack of consistency

The Cross Number Puzzle at the end of page 4 doesn't follow the same pattern as it did before. When "doubling" seemed to be the rule, the number noted in each cell was "doubled." One might expect that an "add 12" would apply to each cell. But the machine simply isn't built to make that response. It evidently wants to add 12 to the check-box number, so it adds 3 to each of the main 4 cell numbers—and thus achieves its result.

Do not expect the kind of consistency you hope for. All you can do is to study the response of the machine and try to understand the way it works so you can predict it reliably.

As the selector is moved from point to point, the machine changes its rule. With each new setting, there's a new game to play.

The first rule on *Work Text* page 5 might be, "multiply by 2 and add 1."

You might explain that you have found a way to help uncover many of the machine rules. It usually multiplies once, or multiplies once and then adds or subtracts. So you make a chart:

Experimenting with the Selector.

The judges next moved the selector to each of the eight positions indicated.

Mr. Wilson kept track of the results, some of which are shown in the chart below. Can you fill in the blanks? Work as far to the right as you can.

Selector position

	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.
No.×2,+													
You write	1	2	5	14	10	9	0	8	50	75	106	500	610
Machine writes	3	5	11	29	21	19	1	17	101	151	213	1001	1221
No.×3,+3													
You write	0	1	2	3	5	7	50	6	44	72	110	314	350
Machine writes	3	6	9	12	18	24	153	21	135	219	333	945	1,053
No.×2,−1													
You write	10	2	7	1	11	3	8	5	15	25	31	200	320
Machine writes	19	3	13	1	21	5	15	9	29	49	61	399	639
No.×6													
You write	3	1	5	2	4	7	6	9	8	20	25	307	515
Machine writes	18	6	30	12	24	42	36	54	48	120	150	1,842	3,090
No.×2													
You write	0	7	1	4	19	36	49	28	100	125	351	161	199
Machine writes	0	14	2	8	38	72	98	56	200	250	702	322	398
No.×4,+1													
You write	2	1	5	10	4	10	9	3	8	100	111	220	213
Machine writes	9	5	21	41	17	41	37	13	33	401	445	881	853
No.×4													
You write	0	11	15	23	13	21	12	14	25	18	23	92	124
Machine writes	0	44	60	92	52	84	48	56	100	72	92	368	496
No.+12													
You write	1	7	8	48	0	29	50	58	88	108	318	490	899
Machine writes	13	19	20	60	12	41	62	70	100	120	330	502	911

When the pointer of Alec Marson's Science Fair project was at "≠," Mr. Wilson tried still another Cross Number Puzzle using some numbers larger than those used in previous examples. Only part of the results are given below. Can you fill in the blanks?

You write here

	add	add	add	add
add →	100	20	7	127
add →	300	70	9	379
add →	400	90	16	506

The machine writes here

400	80	28	508
1200	280	36	1,516
1,600	360	64	2,024

5

you write	1	2	5	14
machine writes	3	5	11	29
2 × what you write	2	4	10	28
3 × what you write	3	6	15	42
4 × what you write	4	8	20	64

You notice that when you multiply by 2 you get a sequence that is 1 less than the machine writes. So, you know one rule might be, "multiply by 2 and add 1 to the result."

Try this approach to the second set of responses on *Work Text* page 5.

you write	0	1	2	3	5
machine writes	3	6	9	12	18
2 × what you write	0	2	4	6	10
3 × what you write	0	3	6	9	15
4 × what you write	0	4	8	12	20

The machine writes 3 more than your second line —the "3 ×" line. So, one rule the machine might be following is, "multiply by 3 and add 3." Of course, you aren't sure. The machine might follow a rule such as, "add 1 and multiply by 3," which is equally as reliable.

This pattern of procedure will help in every example except the last, which might be simply, "add 12."

Rules for the other examples might be:

− "multiply by 2 and then add 1"
+ "multiply by 6"
# "multiply by 2"
= "multiply by 4 and add 1"
≠ "multiply by 4"
"add 12"

Remember, however, that many other procedures will produce equivalent relationships.

Notes to accompany *Work Text* pages 6-9

A similar idea underlies the development in pages 6, 7, 8 and 9. Page 6 might be called "adding a pair of Cross Number Puzzles to give a third puzzle." Page 7 might be called "subtracting one puzzle from another to get a third puzzle." Page 8 might be called "adding a pair of magic squares to get a third magic square." Page 9 might be called "multiplying a Cross Number Puzzle by a number to get another Cross Number Puzzle."

However, don't call them that. Let the patterns emerge so that children are following regularities they notice rather than trying to understand what such verbalization might possibly mean.

It happens that we can precisely define "addition of Cross Number Puzzles" so the puzzles would obey all the rules of numbers. No such definition is available for multiplication of puzzles (for subtle mathematical reasons), but we could find a satisfactory definition for "multiplying a puzzle by a number" as on *Work Text* page 9. However, we do not want to get into complicated definitions.

Let the patterns control the activity. Enough is given in each example to get the idea across.

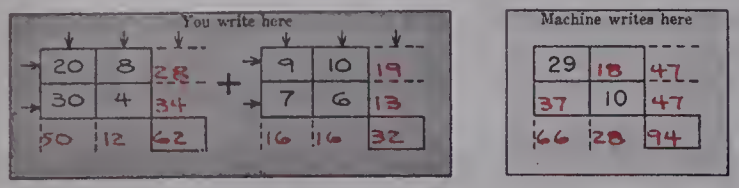
On page 6, the example speaks very clearly. Consider the upper left-hand corner cells in all 3 puzzles. We can express what we see as 20 and 9 go to 29:

6 Mr. Wilson Does His Homework

The judges decided to adjourn until the next morning. Tired, but still curious, they left.

Mr. Wilson, the principal, was so curious, however, that when he left he took Alec's machine with him. He wanted to find out about those "Change Buttons."

In his study at home, he pushed the triangular button. Two Cross Number Puzzle forms flashed on the green screen and one on the blue screen. Mr. Wilson went to work.



Can you see a pattern that suggests a way to fill in the blanks?

Mr. Wilson continued his experiments. Below are shown some of the records he kept:

The green frames appeared on the green screen — the blue frames on the blue screen. Sometimes, Mr. Wilson used a vertical form.

A.

51	13	64	+ 50	8	58	= 101	21	122
20	16	36	30	7	37	50	23	73
71	29	100	80	15	95	151	44	195

B.

11	30	41
10	7	17
21	37	58
+		
8	25	33
15	11	26
23	36	59
=		
19	55	74
25	18	43
44	73	117

C.

100	24	124
200	16	216
300	40	340
+		
300	51	351
100	58	158
400	109	509
=		
400	75	475
300	74	374
700	149	849

D.

45	350	395
32	110	142
77	460	537
+		
29	125	154
28	30	58
57	155	212
=		
74	475	549
60	140	200
134	615	749

7

After completing the five experiments on page 6, Mr. Wilson noticed that the Selector pointer was set at "|." He noted this as one stroke down.

"I wonder what will happen if I move the pointer over to two-strokes-down, (||), he said to himself. He left the triangular button pressed down.

At once, two Cross Number Puzzle forms appeared on the green screen, with a minus sign between them.

M.

5	10	15	− 2	1	3	= 3	9	12
20	15	35	4	3	7	16	12	28
25	25	50	6	4	10	19	21	40

N.

22	8	30	− 3	7	10	= 19	1	20
15	39	54	9	12	21	6	27	33
37	47	84	12	19	31	25	28	53

O.

10	20	30	− 6	2	8	= 4	18	22
30	40	70	4	8	12	26	32	58
40	60	100	10	10	20	30	50	80

P.

33	30	63
20	17	37
53	47	100
−		
10	12	22
11	13	24
21	25	46
=		
23	18	41
9	4	13
32	22	54

Q.

300	42	342
500	49	549
800	91	891
−		
200	17	217
200	22	222
400	39	439
=		
100	25	125
300	27	327
400	52	452

R.

150	26	176
325	80	405
475	106	581
−		
90	7	97
35	36	71
125	43	168
=		
60	19	79
290	44	334
350	63	413

20, 9 $\xrightarrow{+}$ 29

The bottom right-hand corners indicate that 4 and 6 go to 10:

4, 6 $\xrightarrow{+}$ 10

As a further clue, we see a plus sign. So we look at corresponding cells and write:

20, 9 $\xrightarrow{+}$ 29

4, 6 $\xrightarrow{+}$ 10

30, 7 $\xrightarrow{+}$ 37

8, 10 $\xrightarrow{+}$ 18

Now, if we complete these three puzzles, we find that the same relationship holds for all corresponding members of the puzzle. We can extend our list:

28, 19 $\xrightarrow{+}$ 47

34, 13 $\xrightarrow{+}$ 47

50, 16 $\xrightarrow{+}$ 66

12, 16 $\xrightarrow{+}$ 28

62, 32 $\xrightarrow{+}$ 94

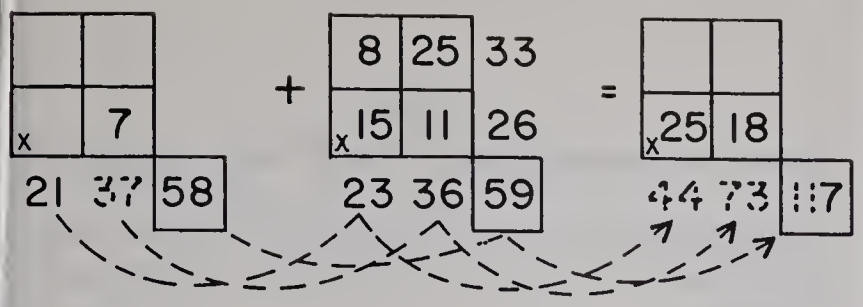
In example A, enough information is given in each of the first two puzzles to complete them. The sum of the top row in the first puzzle must be 100 − 35 or 64. The sum of the center column must be 100 − 71 or 29. In the cell above 29, 16 must be written because 29 − 13 = 16. The relationship outlined here is repeated in the information given:

13, 8 $\xrightarrow{+}$ 21

71, 80 $\xrightarrow{+}$ 151

And the third puzzle can be completed by extending this relationship.

Examples B, C and D are the same, except they are arranged in vertical order. The second puzzle under B contains enough information to permit completion without further clues. Now we must scratch for information. Our problem looks like this:



We know that the sum of the center column in this first puzzle must be 58 − 21 or 37. Now we have enough to compile the row with the check boxes.

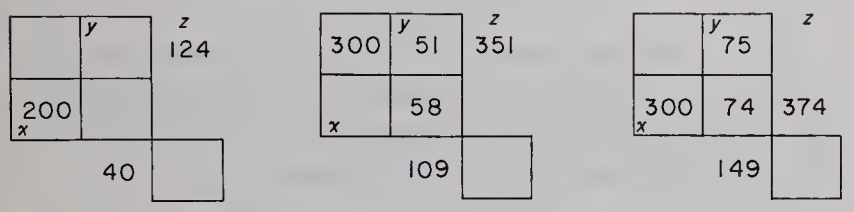
Consider next the cells marked with an "x." We can state the situation as:

$$\_\_\_, 15 \xrightarrow{+} 25$$

And, this leads us to the conclusion that 10 must be written in the cell marked "x" in the first puzzle.

We now have enough given to reconstruct each puzzle by itself. Then we can check whether the relationship we have noticed holds for all pairs of corresponding cells in the first two puzzles and the corresponding cell of the third.

Consider example C. No puzzle can be completely reconstructed with the given information. After the obvious extensions, the problem looks like this:



The relationship of the corresponding cells marked "x" and those marked "y" and those marked "z" is:

"x" 200, ___ $\xrightarrow{+}$ 300

"y" ___, 51 $\xrightarrow{+}$ 75

"z" 124, 351 $\xrightarrow{+}$ ___

So, the cell marked "x" in the second puzzle must be 100, since 200 + 100 = 300. The cell marked "y"

8 "Most interesting and original," Mr. Wilson thought. "I wish I could see inside Alec's machine."

Next, he moved the selector pointer to one-stroke-across — Magic Squares appeared where Cross Number Puzzle forms had been.

A			
4	3	8	15
9	5	1	15
2	7	6	15
15	15	15	15

+

B			
9	7	17	33
19	11	3	33
5	15	13	33
33	33	33	33

=

A + B			
13	10	25	48
28	16	4	48
7	22	19	48
48	48	48	48

C			
16	2	12	30
6	10	14	30
8	18	4	30
30	30	30	30

+

D			
40	5	30	75
15	25	35	75
20	45	10	75
75	75	75	75

=

C + D			
56	7	42	105
21	35	49	105
28	63	14	105
105	105	105	105

E			
6	27	12	45
21	15	9	45
18	3	24	45
45	45	45	45

+

F			
8	36	16	60
28	20	12	60
24	4	32	60
60	60	60	60

=

E + F			
14	63	28	105
49	35	21	105
42	7	56	105
105	105	105	105

G			
12	7	8	27
5	9	13	27
10	11	6	27
27	27	27	27

+

H			
27	22	23	72
20	24	28	72
25	26	21	72
72	72	72	72

=

G + H			
39	29	31	99
25	33	41	99
35	37	27	99
99	99	99	99

Mr. Wilson wondered to himself: "Can you add magic squares in much the same way you can add numbers?"

9 Mr. Wilson was very tired by now, but he was too fascinated by Alec's machine to stop. He moved the selector to one - stroke - across and one - stroke - down.

"Ah-hah!" he said aloud, "it's multiplication — and in the form of Cross Number Puzzles."

a. 2 X

100	30	4	134
300	80	5	385
100	40	6	146
500	150	15	665

=

200	60	8	268
600	160	10	770
200	80	12	292
1,000	300	30	1,330

b. 5 X

100	90	9	199
100	20	3	123
100	50	6	156
300	160	18	478

=

500	450	45	995
500	100	15	615
500	250	30	780
1,500	800	90	2,390

c. 3 X

200	40	5	245
100	20	8	128
300	60	2	362
600	120	15	735

=

600	120	15	735
300	60	24	384
900	180	6	1,086
1,800	360	45	2,205

d. 6 X

200	20	9	229
300	60	2	362
200	60	5	265
700	140	16	856

=

1,200	120	54	1,374
1,800	360	12	2,172
1,200	360	30	1,590
4,200	840	96	5,136

e. 4 X

1,000	70	8	1,078
100	00	4	104
400	40	9	449
1,500	110	21	1,631

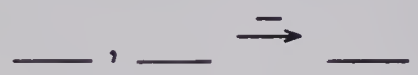
=

4,000	280	32	4,312
400	00	16	416
1,600	160	36	1,796
6,000	440	84	6,524

in the first puzzle must contain 24, since 24 + 51 = 75. The cell marked "z" in the third puzzle must be 475, since 124 + 351 = 475.

That provides sufficient information to reconstruct all three puzzles completely. Then check to see that the relationship holds for all groups of three corresponding cells. Example D will give up its secrets with a similar line of reasoning and computation.

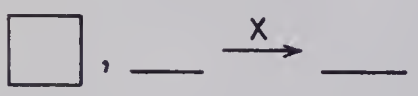
We have gone into considerable detail here because an analogous method may be used on *Work Text* pages 6, 7, 8 and 9. The relationship on page 7 can be written in the form:



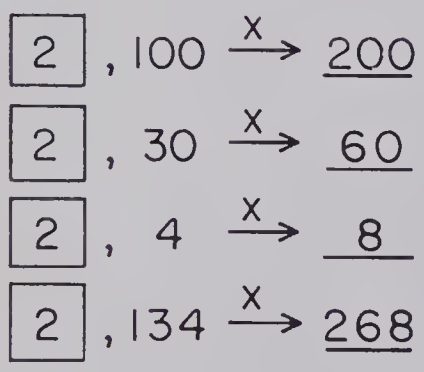
The number noted in a cell in the second puzzle is subtracted from the number noted in the corresponding cell in the first puzzle to lead us to the number to be noted in the corresponding cell of the third puzzle.

On page 8, the relationship is the same as on page 6:

On page 9, the relationship might be expressed as:



In each example, a particular number is written in the box for each example. The top row of correspondencies in example a on page 9 would, when completed, read:



We are suggesting one particular method of attacking these four pages. You may prefer another method. If encouraged, children will invent some of their own—and they ought to be encouraged whenever feasible.

Notes to accompany *Work Text* page 10

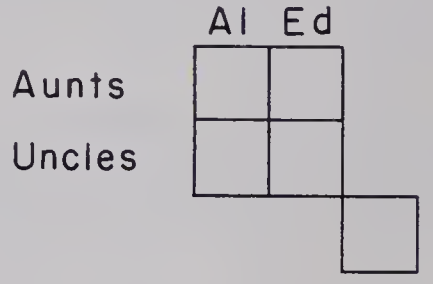
Remember—the next two pages deal with Mr. Wilson's dreams. You must be ready for a high degree of novelty.

"Who ever saw anything in arithmetic like this before?"

"Uncles, aunts, triangles, squares, circles, and lots more—all mixed up!"

Warn the children that they are going to get a glimpse of Mr. Wilson's dreams—and they may seemed mixed up. "Let's work at it together. Maybe we can unscramble the whole thing."

First, it seems that Mr. Wilson dreams of a Cross Number Puzzle form in which he is going to record some information. Put the form on the board:

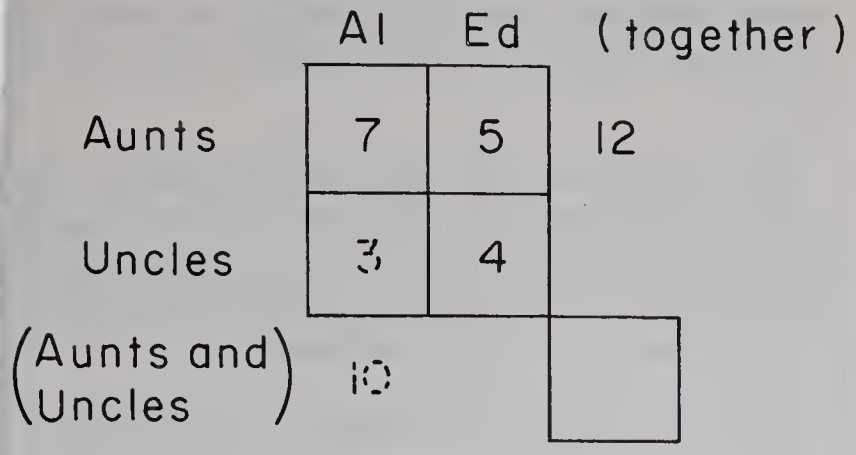


Now read the dream. "Al and Ed were not related." What does that mean? Well, it means that none of Al's uncles or aunts are also Ed's uncles or aunts. There can be no duplication.

Next: "Al had 7 aunts." Where can we record this information in the puzzle? In the column under "Al" and across from "Aunts" in the upper left-hand corner.

Next: "Ed had 4 uncles." Record it under "Ed" and opposite "Uncles."

Next: "Together they had 12 aunts." We know that Al has 7 aunts, and together they have 12 aunts, so Ed must have 5 aunts—because 12 − 7 = 5. Record that under "Ed" and opposite "Aunts." And you can record 12 out to the right—and the column might be titled "together." Now our record looks like this:



Next: "Altogether, Al had 10 relatives that were either uncles or aunts." Well, 7 of these 10 were aunts, so Al must have 3 uncles. Record it under "Al" and across from "Uncles." In the bottom row, record 10—Ed's aunts and uncles together.

Our information ends!

We can supply more by looking at the record. "Ed had 5 aunts and 4 uncles, so that is a total of 9 relatives that were either aunts or uncles. Al and Ed together had 7 uncles." Record this information in the puzzle form.

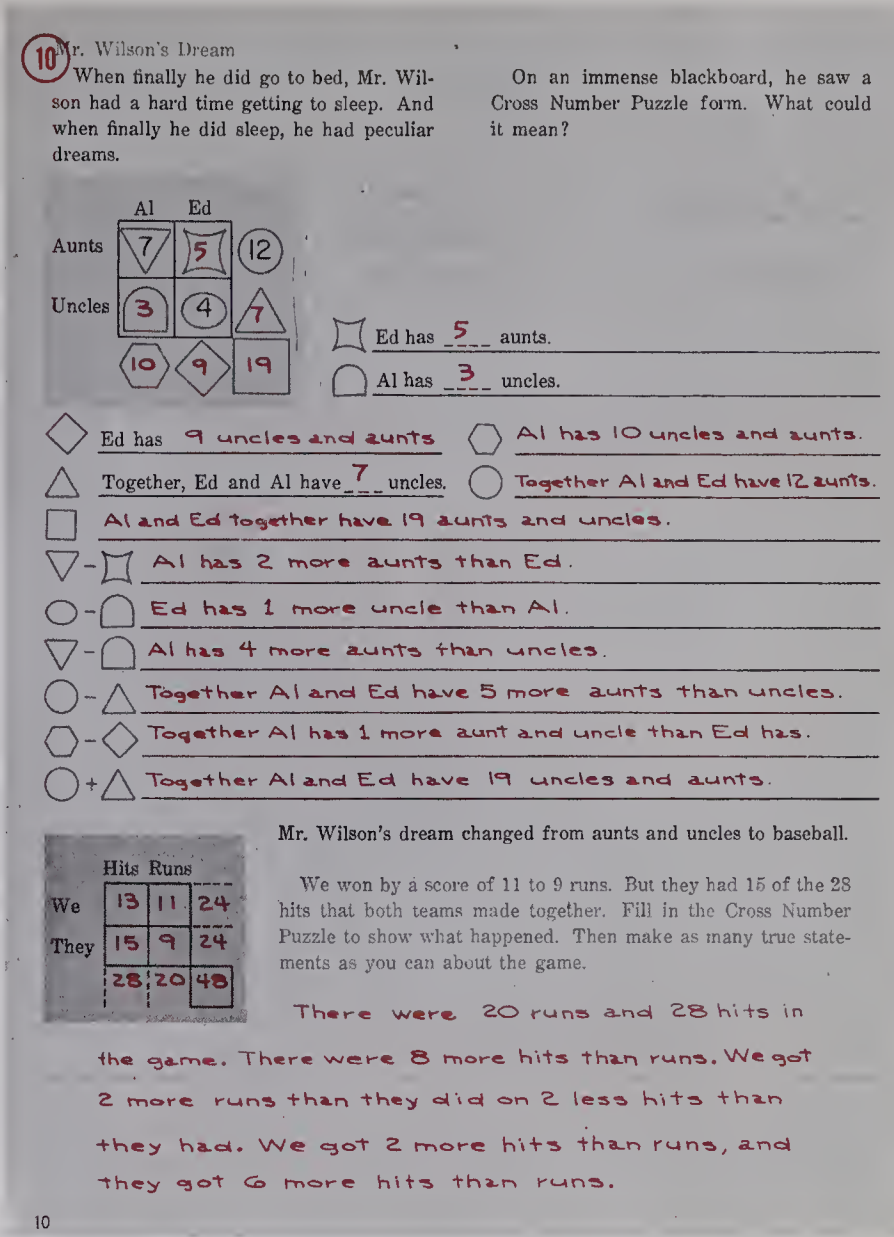
We are almost compelled to put 19 in the check box. But, "What does 19 tell us?" Altogether, Al and Ed had a total of 19 relatives that were either aunts or uncles!

In his dream, Mr. Wilson saw these results written in different shapes. In the work that follows, it is necessary to refer to those shapes to know what Mr. Wilson is talking about.

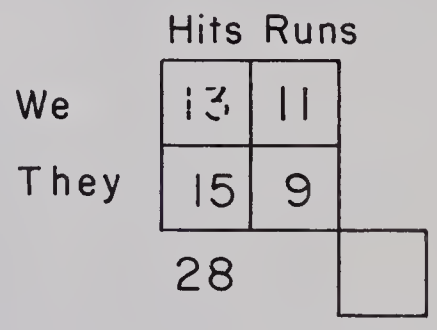
The square is below "Ed" and opposite "Aunts," so it must refer to the number of Ed's relatives that were his aunts . . . 5. The wedge is below "Al" and opposite "Aunts"—and must refer to the number of Al's aunts . . . 7. The wedge minus the square must refer to the difference in the number of aunts each boy counted among his relatives: $7 - 5 = 2$. So we can say that Mr. Wilson's symbols must refer to the fact that Al had 2 more aunts than Ed had. Thus, the first part of the dream begins to unravel.

Then the dream changes abruptly to baseball . . . but keeps the puzzle form as a kind of record blank.

"We won by a score of 11 runs to 9 runs." Record the number of our runs under "runs" and opposite "we." Record the number of their runs under "runs" and opposite "they."


10 Mr. Wilson's Dream
When finally he did go to bed, Mr. Wilson had a hard time getting to sleep. And when finally he did sleep, he had peculiar dreams.
On an immense blackboard, he saw a Cross Number Puzzle form. What could it mean?
Al Ed
Aunts 7 5 12
Uncles 3 4 7
10 9 19
Ed has 5 aunts.
Al has 3 uncles.
Ed has 9 uncles and aunts
Al has 10 uncles and aunts.
Together, Ed and Al have 7 uncles.
Together Al and Ed have 12 aunts.
Al and Ed together have 19 aunts and uncles.
Al has 2 more aunts than Ed.
Ed has 1 more uncle than Al.
Al has 4 more aunts than uncles.
Together Al and Ed have 5 more aunts than uncles.
Together Al has 1 more aunt and uncle than Ed has.
Together Al and Ed have 19 uncles and aunts.
Hits Runs
We 13 11 24
They 15 9 24
28 20 48
Mr. Wilson's dream changed from aunts and uncles to baseball.
We won by a score of 11 to 9 runs. But they had 15 of the 28 hits that both teams made together. Fill in the Cross Number Puzzle to show what happened. Then make as many true statements as you can about the game.
There were 20 runs and 28 hits in the game. There were 8 more hits than runs. We got 2 more runs than they did on 2 less hits than they had. We got 2 more hits than runs, and they got 6 more hits than runs.
10

Next, "They had 15 of the total of 28 hits both teams made together." So, we must have made 13 hits, because $28 - 15 = 13$ or $13 + 15 = 28$. Record this information in the puzzle form:



Obviously, there was a total of 20 runs.

The sum of the number of hits we made and the number of runs is 24. The same statement is true for the other team. There are 8 more hits than runs, etc.

Notes to accompany *Work Text* page 11

Remember—Mr. Wilson is still in the strange land of dreams. And Mrs. Wilson is trying to decide what outfit to wear. It seems she has 4 blue blouses and 2 green blouses; and she has 3 blue skirts and 5

11 Mr. Wilson Dreams On.

Mr. Wilson dreamed of his wife selecting an outfit for a shopping trip.

Blouses: Total	Blue	Green	
6	4	2	Skirts
18	12	6	3 Blue
30	20	10	5 Green
48	32	16	8 Total

She wanted to wear blue or green or both. She was counting her blouses and skirts. She counted out 2 green blouses and 5 green skirts. She also counted out a total of 6 blouses and 8 skirts.

"That's all the information I need," Mr. Wilson said in his dream. "I know how many combinations there are to choose from. The only thing I don't know is which of the outfits she will choose."

Can you complete the chart that Mr. Wilson completed?

a. How many green blouses has she? 2

b. How many blue blouses? 4

c. How many blouses and skirts has she altogether from which to choose? 14

d. How many different outfits can she wear that are all blue? 12

e. How many outfits are green? 10

f. How many with a green blouse? 16

g. How many with a blue blouse? 32

h. How many with a green skirt? 30

i. How many with a blue skirt? 18

j. How many outfits would be all green or all blue? 22

k. How many not all green or all blue? 26

l. How many outfits had she to select from all together? 48.

What ideas are suggested below?

A. △ With a green skirt, she can make 30 outfits.

B. +⬡ She can make a total of 48 outfits.

C. □ She has 48 outfits from which to select.

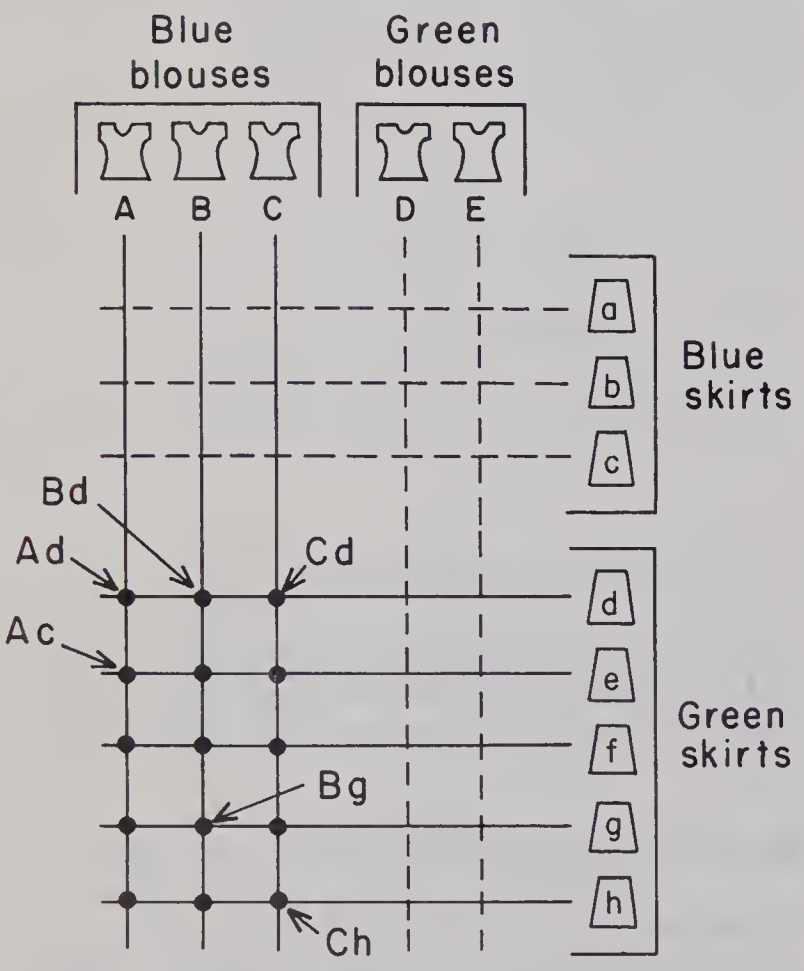
D. ⧈ She has a total of 6 blouses that are green or blue.

E. △ − She has 12 more green skirt outfits than blue ones.

F. −⬡ She has 16 more blue blouse outfits than green ones.

11

green skirts. "If she tries all possible combinations, how many will she try?" A sketch may help:

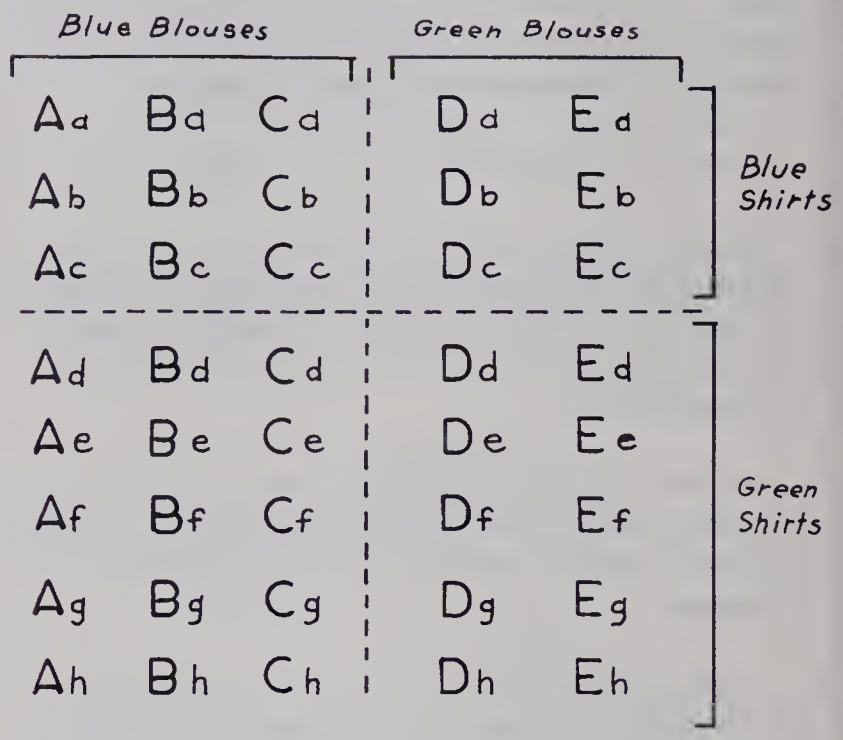


First, let's study the possible combination of a blue blouse and a green skirt. Draw lines down from the blouses that have been lettered A, B and C so we can talk about them. Also, letter the skirts and draw lines across from d, e, f, g and h—all the green skirts.

Mark each place the lines cross. Each of the marks will represent a combination. From the mark, look up to see which blouse is involved; look out to the side and see what skirt is involved. Name each combination as the sketch indicates.

We can be sure there are exactly 15 combinations.

Next, do the same with each possible combination. A summary of the results will be as follows:


Blue Blouses
Green Blouses
Ad Bd Cd Dd Ed
Ab Bb Cb Db Eb
Ac Bc Cc Dc Ec
Blue Shirts
Ad Bd Cd Dd Ed
Ae Be Ce De Ee
Af Bf Cf Df Ef
Ag Bg Cg Dg Eg
Ah Bh Ch Dh Eh
Green Shirts

Now, the plan of his dream is clear. The Cross Number Puzzle is now clearly a record of the combinations Mrs. Wilson could try.

Notes to accompany *Work Text* pages 12-13

Back from dreams to magic squares. Each magic square on pages 12 and 13 will be based on nine different numbers; and the sums of the numbers listed in each row, each column, or each diagonal will name the same number.

Above each example, the magic number will be given, or it will be filled in as soon as it is known. Since most puzzles depend on others, this will often be of great assistance. For example, A(15) means we are referring to example A in which the magic sum is 15.

Put examples A, B, C, D and E on the board.

A (15)

8		
3	5	
		2

Without knowing that the magic sum is 15, you would have a hard time knowing how to proceed. However, if the sum of the diagonal is 15, and two cells are already filled with 5 and 2, the top corner requires 8 because $2 + 5 + 8 = 15$ or $15 - (2 + 5) = 15 - 7 = 8$.

The center row asks for completion of the sentence: $3 + 5 + \_\_\_\_ = 15$ . . . etc. In example B, the magic number is not given, but it can be found: $15 + 18 + 3 = 36$, and the rest is computation. Examples C, D and E fall to similar reasoning and computation. Example Z can be solved easily. One of its diagonals reveals its magic number: $29 + 17 + 5 = 51$.

But call attention to the notation above example Z. What could it mean?

A( ) + B( ) = Z( )

Read, "The magic number of A plus the magic number of B equals the magic number of Z."

Since we knew the magic number of A and B above, we could have found the magic number of Z in another way:

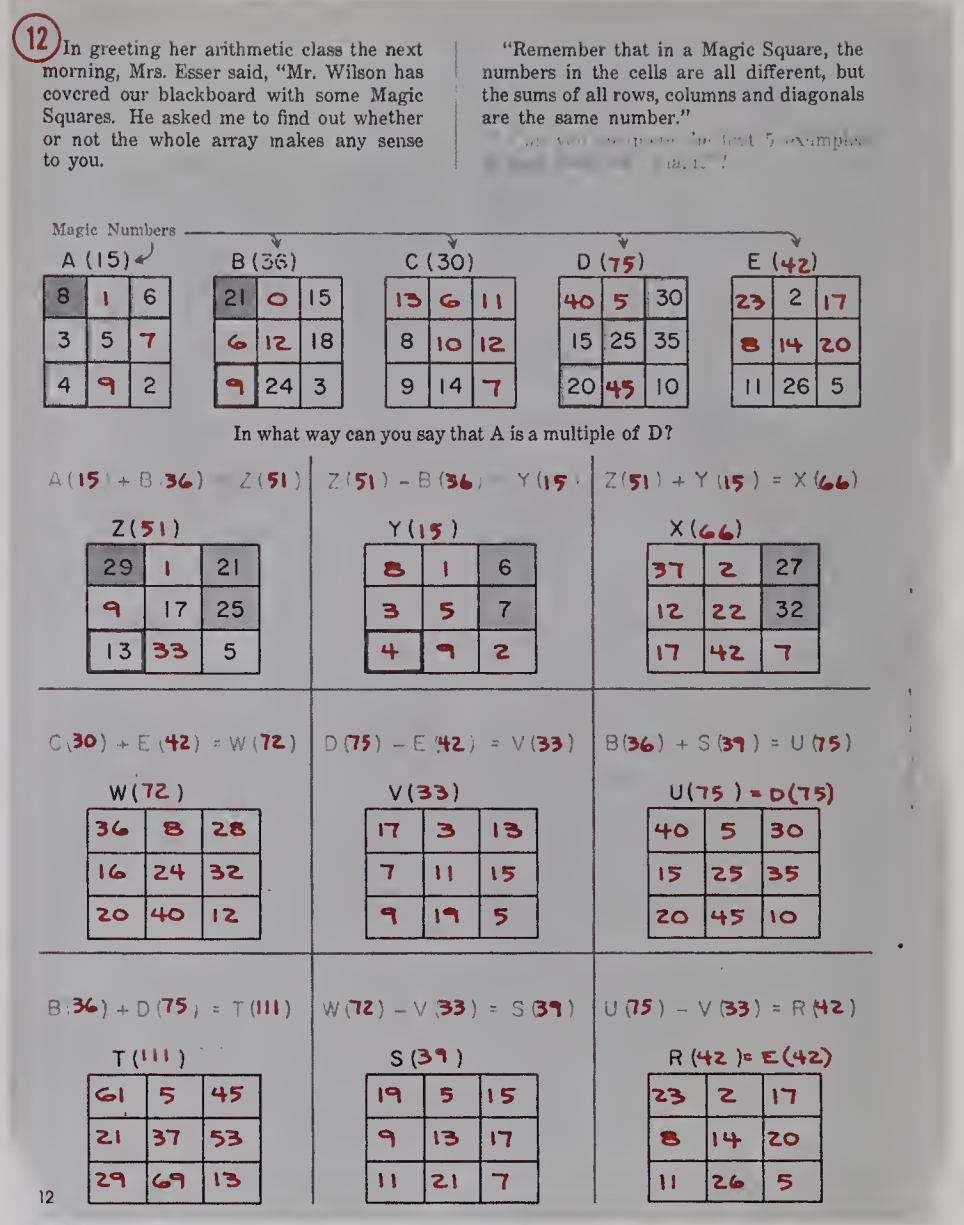
A(15) + B(36) = Z( )

We can compute that: $15 + 36 = 51$.

Now a very important relationship ought to be pointed out: Analysis will reveal that the sum of numbers noted in corresponding cells of A and B are written in the corresponding cell of Z. For example, in the upper left-hand corner of A and B are 8 and 21, and in the corresponding cell of Z is 29. This relationship holds throughout. A simple color-coding scheme helps reveal this relationship.

Above example Y is the expression:

Z( ) − B( ) = Y( )


12

In greeting her arithmetic class the next morning, Mrs. Esser said, "Mr. Wilson has covered our blackboard with some Magic Squares. He asked me to find out whether or not the whole array makes any sense to you.

"Remember that in a Magic Square, the numbers in the cells are all different, but the sums of all rows, columns and diagonals are the same number."

Magic Numbers

A (15)

8	1	6
3	5	7
4	9	2

B (36)

21	0	15
6	12	18
9	24	3

C (30)

13	6	11
8	10	12
9	14	7

D (75)

40	5	30
15	25	35
20	45	10

E (42)

23	2	17
8	14	20
11	26	5

In what way can you say that A is a multiple of D?

A(15) + B(36) = Z(51)

Z(51)

29	1	21
9	17	25
13	33	5

Z(51) − B(36) = Y(15)

Y(15)

8	1	6
3	5	7
4	9	2

Z(51) + Y(15) = X(66)

X(66)

37	2	27
12	22	32
17	42	7

C(30) + E(42) = W(72)

W(72)

36	8	28
16	24	32
20	40	12

D(75) − E(42) = V(33)

V(33)

17	3	13
7	11	15
9	19	5

B(36) + S(39) = U(75)

U(75) = D(75)

40	5	30
15	25	35
20	45	10

B(36) + D(75) = T(111)

T(111)

61	5	45
21	37	53
29	69	13

W(72) − V(33) = S(39)

S(39)

19	5	15
9	13	17
11	21	7

U(75) − V(33) = R(42)

R(42) = E(42)

23	2	17
8	14	20
11	26	5

12

The magic number of Z (which is 51) minus the magic number of B (which is 36) equals the magic number of Y, and $51 - 36 = 15$.

Now subtract each cell of B from the corresponding cell of Z to find a number for the corresponding cell in Y.

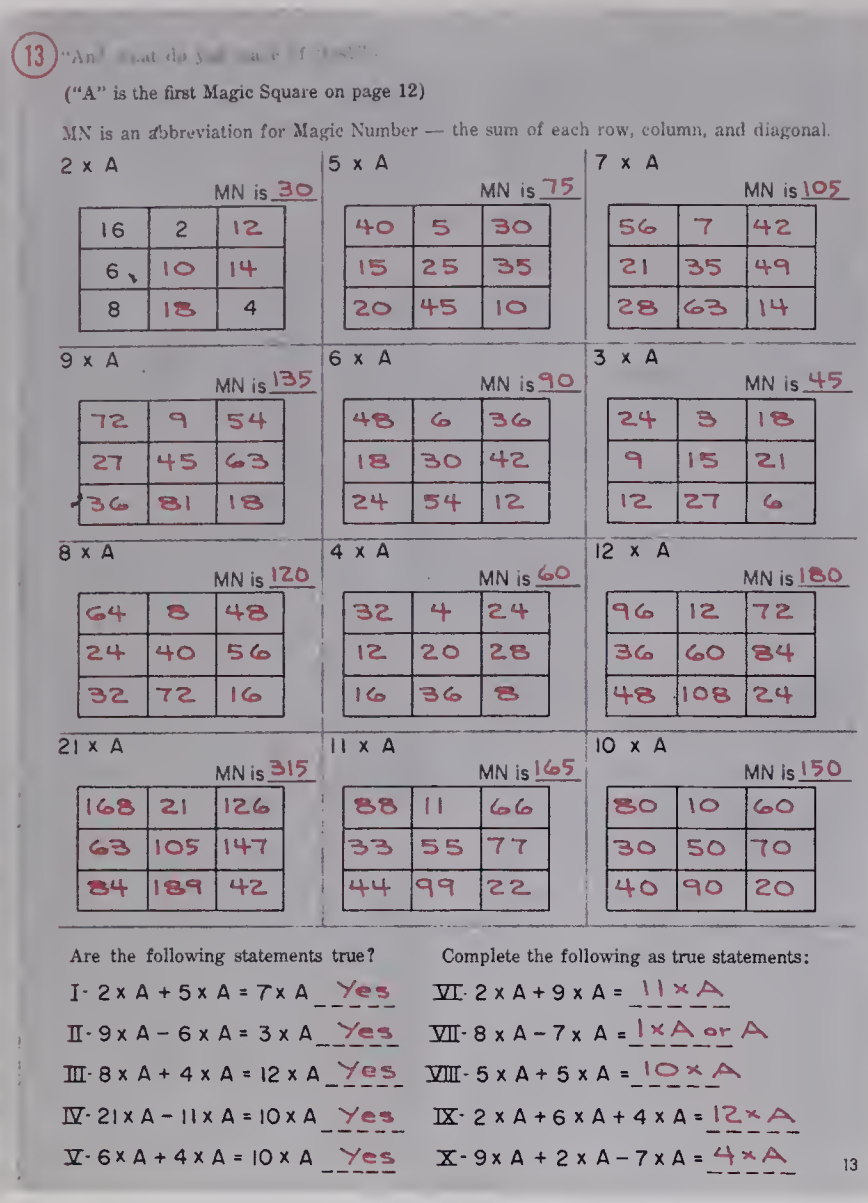
Z	B		Y
29	21	→	8
1	0	→	1
21	15	→	6
9	6	→	3

And when the Y square is completed, we can write:

Y(15) = A(15)

Further inspection shows them to be identical magic squares.

In the remainder of the examples on *Work Text* page 12, there is, as in example Y, dependence on


13 "An[illegible]"

("A" is the first Magic Square on page 12)

MN is an abbreviation for Magic Number — the sum of each row, column, and diagonal.

2 x A — MN is 30

16	2	12
6	10	14
8	18	4

5 x A — MN is 75

40	5	30
15	25	35
20	45	10

7 x A — MN is 105

56	7	42
21	35	49
28	63	14

9 x A — MN is 135

72	9	54
27	45	63
36	81	18

6 x A — MN is 90

48	6	36
18	30	42
24	54	12

3 x A — MN is 45

24	3	18
9	15	21
12	27	6

8 x A — MN is 120

64	8	48
24	40	56
32	72	16

4 x A — MN is 60

32	4	24
12	20	28
16	36	8

12 x A — MN is 180

96	12	72
36	60	84
48	108	24

21 x A — MN is 315

168	21	126
63	105	147
84	189	42

11 x A — MN is 165

88	11	66
33	55	77
44	99	22

10 x A — MN is 150

80	10	60
30	50	70
40	90	20

Are the following statements true?

I- 2 x A + 5 x A = 7 x A Yes

II- 9 x A − 6 x A = 3 x A Yes

III- 8 x A + 4 x A = 12 x A Yes

IV- 21 x A − 11 x A = 10 x A Yes

V- 6 x A + 4 x A = 10 x A Yes

Complete the following as true statements:

VI- 2 x A + 9 x A = 11 x A

VII- 8 x A − 7 x A = 1 x A or A

VIII- 5 x A + 5 x A = 10 x A

IX- 2 x A + 6 x A + 4 x A = 12 x A

X- 9 x A + 2 x A − 7 x A = 4 x A

13

magic squares already completed. The key to this dependence is the expression written above each example.

In examples on page 13, references are made to the first example, A, on page 12. "MN" is the abbreviation for "magic number." "2 × A" is the abbreviation for "find the number of the new square by multiplying each number by 2 that is noted in example A on page 12—and write the result in its corresponding cell of the new square." (Refer to the activity on *Work Text* page 8.)

Each cell of the new square is twice the corresponding cell of A on the previous page. The activity on this page requires as entries all facts in the multiplication tables of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and 21.

At the bottom of this page, five completed statements are given. The first is: $2 \times A + 5 \times A = 7 \times A$. True or false? If numbers noted in the corresponding cells of "2 × A" and "5 × A" are added, is the sum written in the corresponding cell of "7 × A?" Yes. Is it also true that: $2 \times A(\quad) + 5 \times A(\quad) = 7 \times A(\quad)$?

Is the magic number of "2 × A" plus the magic number of "5 × A" the same as the magic number of "7 × A?" (Again, yes.) The magic quality of these squares has been preserved under all kinds of manipulations.

Perhaps some children would like to experiment with other manipulations to see under what conditions the magic quality is preserved. (It is not always preserved. If the rule is "square each number," then we have:

4	3	8
9	5	1
2	7	6

→

16	9	64
81	25	1
4	49	36

which is clearly not "magic," at least in the sense we have been using that word.)

8	1	6
3	5	7
4	9	2

+

13	12	17
18	14	10
11	16	15

→

21	13	23
21	19	17
15	25	17

Something happened! Why is the third square no longer a magic square? The sum of rows, columns, and diagonals meets the test, but 21 and 17 are repeated. Magic squares have intrigued mathematicians for centuries. They are still fascinating—to young and old alike.

Notes to accompany *Work Text* pages 14-17

Pages 14, 15, 16 and 17 start out in a game-like atmosphere. Can you predict what Alec's machine is going to write?

For the teacher alone

Each activity has a lot to do with division—but that's not all. *Nothing* ought to be said about it now, but Alec's machine is able to add, subtract, multiply, divide fractions.

These four pages are concerned only with such expressions as 12/3, which is another way of writing $3\overline{)12}$—"12 divided by 3" in both cases—and, in the four pages, expressions are limited to expressions that name whole numbers. Eventually, on *Work Text* page 56, we will again reconsider this particular machine and find that it can handle such expressions as ⅔ and 12/5, adding, subtracting, mul-

tiplying and dividing any fractions. But remember, the less said about this whole matter at this time, the better.

For the children

Another of Alec's machines. Mr. Wilson's experiments helped him understand that, in this case, he needed to write names for two whole numbers, each in the form of one number divided by another.

$12 \div 2$, $3\overline{)15}$ and 24/8 were acceptable to the machine.

$12 \div 12$, $3\overline{)16}$ and 8/24 were not acceptable.

But the machine's responses were certainly puzzling. It gives four separate responses, each seeming to follow a different rule—A, B, C and D. Consider these rules one at a time.

Rule A

15/1, 3/1 → 18/1

20/2, 2/1 → 24/2

20/2, 5/1 → 30/2

12/2, 6/1 → 24/2

20/5, 4/2 → 60/10

The relationship of the numbers written below the line—divisor or denominator—is clear.

1, 1 → 1

2, 1 → 2 (repeated 3 times)

5, 2 → 10

Check the rest of the example under column A on page 14. Does the rule hold in every case? (Yes!)

A similar approach to the numbers written above the lines doesn't help at all.

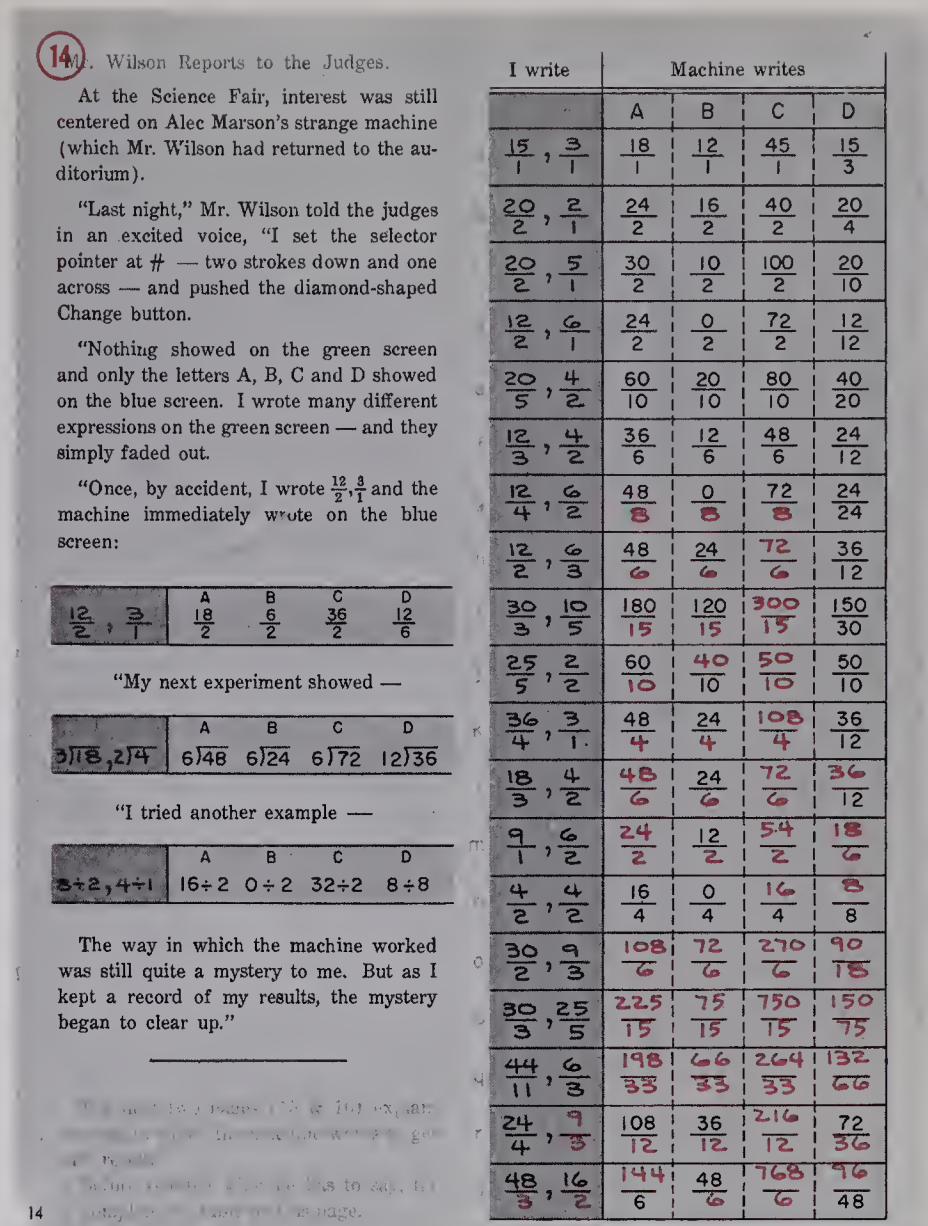
15, 3 → 18

20, 2 → 24

20, 5 → 30

12, 6 → 24

20, 4 → 60



14 Mr. Wilson Reports to the Judges.

At the Science Fair, interest was still centered on Alec Marson's strange machine (which Mr. Wilson had returned to the auditorium).

"Last night," Mr. Wilson told the judges in an excited voice, "I set the selector pointer at # — two strokes down and one across — and pushed the diamond-shaped Change button.

"Nothing showed on the green screen and only the letters A, B, C and D showed on the blue screen. I wrote many different expressions on the green screen — and they simply faded out.

"Once, by accident, I wrote $\frac{12}{2}, \frac{3}{1}$ and the machine immediately wrote on the blue screen:

	A	B	C	D
$\frac{12}{2}, \frac{3}{1}$	$\frac{18}{2}$	$\frac{6}{2}$	$\frac{36}{2}$	$\frac{12}{6}$

"My next experiment showed —

	A	B	C	D
$3\overline{)18}, 2\overline{)4}$	$6\overline{)48}$	$6\overline{)24}$	$6\overline{)72}$	$12\overline{)36}$

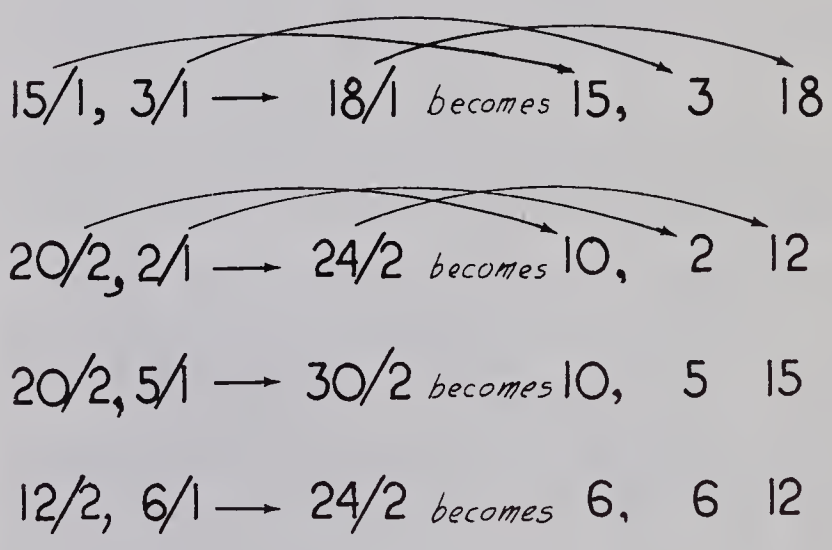
"I tried another example —

	A	B	C	D
8÷2, 4÷1	16÷2	0÷2	32÷2	8÷8

The way in which the machine worked was still quite a mystery to me. But as I kept a record of my results, the mystery began to clear up."

I write	Machine writes			
	A	B	C	D
$\frac{15}{1}, \frac{3}{1}$	$\frac{18}{1}$	$\frac{12}{1}$	$\frac{45}{1}$	$\frac{15}{3}$
$\frac{20}{2}, \frac{2}{1}$	$\frac{24}{2}$	$\frac{16}{2}$	$\frac{40}{2}$	$\frac{20}{4}$
$\frac{20}{2}, \frac{5}{1}$	$\frac{30}{2}$	$\frac{10}{2}$	$\frac{100}{2}$	$\frac{20}{10}$
$\frac{12}{2}, \frac{6}{1}$	$\frac{24}{2}$	$\frac{0}{2}$	$\frac{72}{2}$	$\frac{12}{12}$
$\frac{20}{5}, \frac{4}{2}$	$\frac{60}{10}$	$\frac{20}{10}$	$\frac{80}{10}$	$\frac{40}{20}$
$\frac{12}{3}, \frac{4}{2}$	$\frac{36}{6}$	$\frac{12}{6}$	$\frac{48}{6}$	$\frac{24}{12}$
$\frac{12}{4}, \frac{6}{2}$	$\frac{48}{8}$	$\frac{0}{8}$	$\frac{72}{8}$	$\frac{24}{24}$
$\frac{12}{2}, \frac{6}{3}$	$\frac{48}{6}$	$\frac{24}{6}$	$\frac{72}{6}$	$\frac{36}{12}$
$\frac{30}{3}, \frac{10}{5}$	$\frac{180}{15}$	$\frac{120}{15}$	$\frac{300}{15}$	$\frac{150}{30}$
$\frac{25}{5}, \frac{2}{2}$	$\frac{60}{10}$	$\frac{40}{10}$	$\frac{50}{10}$	$\frac{50}{10}$
$\frac{36}{4}, \frac{3}{1}$	$\frac{48}{4}$	$\frac{24}{4}$	$\frac{108}{4}$	$\frac{36}{12}$
$\frac{18}{3}, \frac{4}{2}$	$\frac{48}{6}$	$\frac{24}{6}$	$\frac{72}{6}$	$\frac{36}{12}$
$\frac{9}{1}, \frac{6}{2}$	$\frac{24}{2}$	$\frac{12}{2}$	$\frac{54}{2}$	$\frac{18}{6}$
$\frac{4}{2}, \frac{4}{2}$	$\frac{16}{4}$	$\frac{0}{4}$	$\frac{16}{4}$	$\frac{8}{8}$
$\frac{30}{2}, \frac{9}{3}$	$\frac{108}{6}$	$\frac{72}{6}$	$\frac{270}{6}$	$\frac{90}{18}$
$\frac{30}{3}, \frac{25}{5}$	$\frac{225}{15}$	$\frac{75}{15}$	$\frac{750}{15}$	$\frac{150}{75}$
$\frac{44}{11}, \frac{6}{3}$	$\frac{198}{33}$	$\frac{66}{33}$	$\frac{264}{33}$	$\frac{132}{66}$
$\frac{24}{4}, \frac{9}{3}$	$\frac{108}{12}$	$\frac{36}{12}$	$\frac{216}{12}$	$\frac{72}{36}$
$\frac{48}{3}, \frac{16}{2}$	$\frac{144}{6}$	$\frac{48}{6}$	$\frac{768}{6}$	$\frac{96}{48}$

Skip to page 15. Mr. Wilson noticed that he could rewrite the examples under A as whole numbers and they make real sense.



So, one approach to the rule might be: Change the example into an expression involving whole numbers and think of addition. Then write the sum as an example of division (or as a fraction) with the divisor (or denominator) according to the pattern seen previously. For instance:

15 "The meaning of A, B, C and D on the blue screen became clear when I rewrote my table of results (shown on page 14) as whole numbers rather than as indicated divisors.

	I write	Machine writes			
		A	B	C	D
o.	15, 3	18	12	45	5
b.	10, 2	12	8	20	5
c.	10, 5	15	5	50	2
d.	6, 6	12	0	36	1
e.	4, 2	6	2	8	2
f.	4, 2	6	2	8	2
g.	3, 3	6	0	9	1
h.	6, 2	8	4	12	3
i.	10, 2	12	8	20	5
j	5, 1	6	4	5	5
k.	9, 3	12	6	27	3
l.	6, 2	8	4	12	3
m.	9, 3	12	6	27	3
n.	2, 2	4	0	4	1
o.	15, 3	18	12	45	5
p.	10, 5	15	5	50	2
q.	4, 2	6	2	8	2
r.	6, 3	9	3	18	2
s.	16, 8	24	8	128	2

"With that mystery solved — A, B, C and D indicating addition, subtraction, multiplication and division, respectively — I wondered how the machine got its particular form of result. It is true that 20 ÷ 2 x 10 ÷ 2 = 200 ÷ 4 (see example below) because 10 x 5 = 50, but why didn't the machine write 50 ÷ 1 as the result; or 100 ÷ 2 or 500 ÷ 10?

"So, I compared my record with the machine's response in 10 of the examples under column C."

		C
a.	15/1, 3/1	45/1
b.	20/2, 2/1	40/2
c.	20/2, 5/1	100/2
d.	12/2, 6/1	72/2
e.	20/5, 4/2	80/10
f.	12/3, 4/2	48/6
g.	12/4, 6/2	72/8
h.	12/2, 6/3	72/6
i.	30/3, 10/5	300/15
j	25/5, 2/2	50/10

"One way the machine would work to get these results became obvious immediately, so I turned to the results in column D."

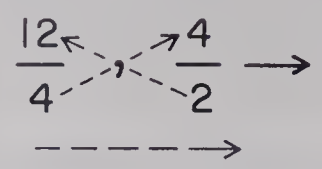
		D
a.	15/1, 3/1	15/3
b.	20/2, 2/1	20/4
c.	20/2, 5/1	20/10
d.	12/2, 6/1	12/12
e.	20/5, 4/2	40/20
f.	12/3, 4/2	24/12
g.	12/4, 6/2	24/24
h.	12/2, 6/3	36/12
i.	30/3, 10/5	150/30
j.	25/5, 2/2	50/10

"One way the machine may work is this:"

I write	Machine works		Machine writes
20/2, 5/1	20 5 / 2 1	1 x 20 = 20 5 x 2 = 10	20/10
12/2, 6/3	12 6 / 2 3	3 x 12 = 36 6 x 2 = 12	36/12

15

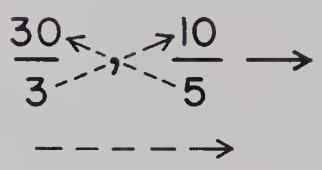
Each example under A will yield to this line of attack. Someone along the way may see a shortcut that will work every time. He might explain it with a sketch like this:



2 x 12 = 24 and 4 x 4 = 16 and 24 + 16 = 40 - - - → 40

4 x 2 = 8 - → 8

The machine will write 48/8. Another example:



5 x 30 = 150 and 3 x 10 = 30 and 150 + 30 = 180 - - → 180

3 x 5 = 15 - → 15

The machine will write 180/15. So we now have two different methods of predicting what the machine will write.

Rule B

This rule will yield its secrets to the same kind of attack. Use the method Mr. Wilson uses on *Work Text* page 15. He converts the fractions or examples of division to whole numbers, and finds this pattern:

15/1 , 3/1 → 12/1 and 15 , 3 → 12

20/2 , 2/1 → 16/2 and 10 , 2 → 8

20/2 , 5/1 → 10/2 and 10 , 5 → 5

12/2 , 6/1 → 0/2 and 6 , 6 → 0

It is quite clear that subtraction gets into the act.

Now, the denominator or divisors seem to follow exactly the same rule as they did in Rule A. So, we work out the fifth example:

12/3, 4/2 → ?

As whole numbers and addition:

12/3, 4/2 → 4, 2 and 4, 2 → 6

Now we also know the machine will use 3 × 2 = 6 to find the divisor or denominator of the new expression, so we list

12/3, 4/2 → ?/6

. . . and the question mark must be replaced by a number so that

—— ÷ 6 = 6 or 36 ÷ 6 = 6

So, the machine will write:

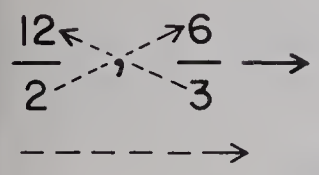
36/6

$\frac{20}{5}, \frac{4}{2} \rightarrow \frac{?}{10}$

$\frac{20}{5}, \frac{4}{2} \rightarrow 4, 2 \rightarrow 2$ and $2 = \frac{20}{10}$

The machine will write 20/10! . . . and it does.

Is there a shortcut? Yes, but do not reveal it straight-away. A few hints will suffice to bring out the shortcut. It is:



3 x 12 = 36 and 2 x 6 = 12 and 36 − 12 = 24 - - - → 24

2 x 3 = 6 - → 6

The machine will write 24/6. Right again.

Rule C

This rule almost reveals itself. The approach that worked for divisors or denominators in Rules A and B—and also for dividends or numerators as well—that's one way to predict the machine.

Rule D

Results of applying this rule can be anticipated by employing the same attack as in cases of Rules A and B. Mr. Wilson does this on page 15, and finds that division is involved. A shortcut in this case is given at the end of page 15.

On *Work Text* page 16, Mr. Wilson explores Rules A and B further, and finds that the methods outlined above are clearly reliable.

On page 17, Judge Harris is a little troubled by the clumsy method the machine seems to use when there are two simpler ways to get equivalent results. Obviously, the machine is adding, subtracting, multiplying and dividing. And Judge Harris is sure Alec knew that in many of the examples he could have used the Least Common Multiple as a denominator, and thus simplify the procedure. He is also disturbed because each expression could be reduced immediately to an expression with 1 as divisor or demoninator. On page 17, several examples are considered in all three ways.

Mr. Wilson reminds the judges that perhaps they don't really know what these rules A, B, C and D were intended to accomplish. Alec may have had something very different in mind. Who is to say that the machine was intended to handle fractions?

16

"The results in the A column gave me more trouble," Mr. Wilson continued. "I made up a pattern or series of examples and noted the following results:"

	I write	Machine writes	I check
A	10/1, 6/1	16/1	10 + 6 = 16
B	10/1, 6/2	26/2	10 + 3 = 13
C	10/1, 6/3	36/3	10 + 2 = 12
D	10/2, 6/1	22/2	5 + 6 = 11
E	10/2, 6/2	32/4	5 + 3 = 8
F	10/2, 6/3	42/6	5 + 2 = 7
G	10/5, 6/1	40/5	2 + 6 = 8
H	10/5, 6/2	50/10	2 + 3 = 5
I	10/5, 6/3	60/15	2 + 2 = 4

"I knew that the results in column B were, in some way, based on subtraction, and I made up a similar set of examples." (You can complete the record.)

	I write	Machine writes	I check
a.	12/1, 6/2	18/2	12 − 3 = 9
b.	12/1, 6/3	30/3	12 − 2 = 10
c.	12/2, 6/2	12/4	6 − 3 = 3
d.	12/2, 6/3	24/6	6 − 2 = 4
e.	12/3, 6/2	6/6	4 − 3 = 1
f.	12/3, 6/3	18/9	4 − 2 = 2
g.	12/4, 6/2	0/8	3 − 3 = 0
h.	12/4, 6/3	12/12	3 − 2 = 1
i.	12/4, 6/6	48/24	3 − 1 = 2

"I saw a pattern," Mr. Wilson went on, "that seemed to lead me to the same results as the machine. So, I tried more examples. Here are the records I kept:"

	I write	Machine writes A	Machine writes B	I check
I	4/2, 2/1	8/2	0/2	A 2 + 2 = 4; B 2 − 2 = 0
II	6/2, 2/1	10/2	2/2	A 3 + 2 = 5; B 3 − 2 = 1
III	8/2, 2/1	12/2	4/2	A 4 + 2 = 6; B 4 − 2 = 2
IV	10/2, 2/1	14/2	6/2	A 5 + 2 = 7; B 5 − 2 = 3
V	6/2, 3/1	12/2	0/2	A 3 + 3 = 6; B 3 − 3 = 0
VI	9/3, 3/1	18/3	0/3	A 3 + 3 = 6; B 3 − 3 = 0
VII	20/5, 3/1	35/5	5/5	A 4 + 3 = 7; B 4 − 3 = 1
VIII	12/2, 6/3	48/6	24/6	A 6 + 2 = 8; B 6 − 2 = 4
IX	5/1, 6/3	21/3	9/3	A 5 + 2 = 7; B 5 − 2 = 3
X	10/5, 6/3	60/15	0/15	A 2 + 2 = 4; B 2 − 2 = 0
XI	9/3, 4/2	30/6	6/6	A 3 + 2 = 5; B 3 − 2 = 1
XII	10/2, 14/7	98/14	42/14	A 5 + 2 = 7; B 5 − 2 = 3

17

"I'm a little troubled," Judge Harris said, "by this development with Alec's machine. It seems to say that

$\frac{10}{2} + \frac{6}{2} = \frac{20}{4} + \frac{12}{4} = \frac{32}{4}$

in example E on page 16. We check by noting that

5 + 3 = 8

and we know that 32 ÷ 4 = 8."

Judge Thornton broke in, "But it seems that Alec invented his machine to work in a way that uses only fractions."

"True," Judge Harris admitted, "but I'm sure Alec knew how to use the least common multiple as the lowest common denominator. Yet his machine seems to say

$\frac{12}{4} - \frac{6}{6} = \frac{72}{24} - \frac{24}{24} = \frac{48}{24}$

Yet, I feel quite sure that Alec would have written

$\frac{12}{4} - \frac{6}{6} = \frac{36}{12} - \frac{12}{12} = \frac{24}{12}$

In fact, if Alec were here, I'm sure he would simplify each of those fractions and write

$\frac{12}{4} - \frac{6}{6} = \frac{3}{1} - \frac{1}{1} = \frac{2}{1}$

Try the following examples in all three ways. I think you will agree with me."

The Machine's Method: Find a common denominator by multiplying the denominators.	Least Common Denominator: Find the least common multiple of the denominators.	Simplify the fractions
12/2 − 6/2 = 24/4 − 12/4 = 12/4	12/2 − 6/2 = 6/2 = 3/1	12/2 − 6/2 = 6/1 − 3/1 = 3/1
12/3 − 6/3 = 36/9 − 18/9 = 18/9	12/3 − 6/3 = 6/3 = 2/1	12/3 − 6/3 = 4/1 − 2/1 = 2/1
8/2 + 12/6 = 48/12 + 24/12 = 72/12	8/2 + 12/6 = 24/6 + 12/6 = 36/6 = 6/1	8/2 + 12/6 = 4/1 + 2/1 = 6/1
20/10 + 15/5 = 100/50 + 150/50 = 250/50	20/10 + 15/5 = 20/10 + 30/10 = 50/10 = 5/1	20/10 + 15/5 = 2/1 + 3/1 = 5/1
4/4 + 14/14 = 64/64 + 64/64 = 128/64	4/4 + 14/14 = 28/28 + 28/28 = 56/28 = 2/1	4/4 + 14/14 = 1/1 + 1/1 = 2/1
24/6 − 12/4 = 96/24 − 72/24 = 24/24	24/6 − 12/4 = 48/12 − 36/12 = 12/12 = 1/1	24/6 − 12/4 = 4/1 − 3/1 = 1/1
24/12 + 8/4 = 96/48 + 96/48 = 192/48	24/12 + 8/4 = 24/12 + 24/12 = 48/12 = 4/1	24/12 + 8/4 = 2/1 + 2/1 = 4/1
8/3 + 36/18 = 108/54 + 108/54 = 216/54	6/3 + 36/18 = 36/18 + 36/18 = 72/18 = 4/1	6/3 + 36/18 = 2/1 + 2/1 = 4/1
30/6 − 16/8 = 240/48 − 96/48 = 144/48	30/6 − 16/8 = 120/24 − 48/24 = 72/24 = 3/1	30/6 − 16/8 = 5/1 − 2/1 = 3/1
16/8 − 12/12 = 192/96 − 96/96 = 96/96	16/8 − 12/12 = 48/24 − 24/24 = 24/24 = 1/1	16/8 − 12/12 = 2/1 − 1/1 = 1/1

Mr. Wilson has an idea

"As I was working these examples," Mr. Wilson said, "it occurred to me that we may be unfair to Alec. He never told us what his machine was designed to do. Perhaps we are using his machine in a way he didn't intend it to be used."

General notes to accompany Sue Marson's games (*Work Text* pages 18-33)

Alec's sister, Sue Marson, entered some ingenious games in the School Science Fair. Let's explore the role these games play in developing the ideas of arithmetic.

Sue's games go to the heart of the idea of systems of numeration—including our decimal system. They reveal, without any verbalization, the vital idea of *place value*. They help develop an understanding of the problems encountered with denominal numbers —in combining measurements, such as adding 2 feet and 7 inches to 1 foot and 9 inches and finding the simplest expression of the total length: 2 feet and 7 inches plus 1 foot and 9 inches is the same as 1 yard, 1 foot and 4 inches.

In the past, some teachers have introduced children to the idea of other bases of numeration—the binary system, the quinary system, the duodecimal system, the systems of numeration to the base 3, to the base 7, etc. These are interesting ideas, but, they lose much of their value in helping with the job at hand when the ideas are introduced as abstract exercises.

"How do you add 3 and 2 in the base 2?" is a typical "abstract" question. The child is expected to write:

$$\begin{array}{rr} & 1\ 1 \\ & \text{two} \\ + & 1\ 0 \\ & \text{two} \\ \hline & 1\ 0\ 1 \\ & \text{two} \end{array}$$

The answer is "101 two"; read, "one, zero, one in the base two."

This kind of activity has led to much misunderstanding among teachers and parents. It seems strange and unrelated to the world we live in. The opposite is true, however.

Let us approach the same ideas in a different way. Question: "How can you buy three glasses of milk at the store in the fewest cartons?" (We specify that cartons available are half-pints, pints and quarts—and that a "glass" is the same as a half-pint.

Quarts	Pints	Half-Pints	
	1	1	3 Glasses

Answer: "One pint and one half-pint."

Question: "How can you buy two glasses of milk at the store in the fewest cartons?"

Quarts	Pints	Half-Pints	
	1	0	2 Glasses

Answer: "One pint carton will be enough."

Question: "Suppose you wanted to buy five glasses all at once. How could you do it in the fewest cartons?"

Quarts	Pints	Half-Pints	
1	0	1	5 Glasses

Answer: "I would need to get only two cartons—one quart and one half-pint—and I would need no pints."

Now, let's put those three results together:

Quarts	Pints	Half-Pints	
	1	1	3 Glasses
	1	0	2 Glasses
1	0	1	5 Glasses

Question: "Instead of buying three glasses of milk at one time in a pint carton and a half-pint carton, and two glasses in a pint carton, suppose you bought the five (3 + 2) glasses at one time in the fewest cartons."

Compare this very practical situation with the answer expected of a child in the earlier question—"three plus two in the base two:"

$$\begin{array}{rrl} & 1\ 1 & \text{(Three written in base two)} \\ & \text{two} & \\ + & 1\ 0 & \text{(Two written in base two)} \\ & \text{two} & \\ \hline & 1\ 0\ 1 & \text{(Five written in base two)} \\ & \text{two} & \end{array}$$

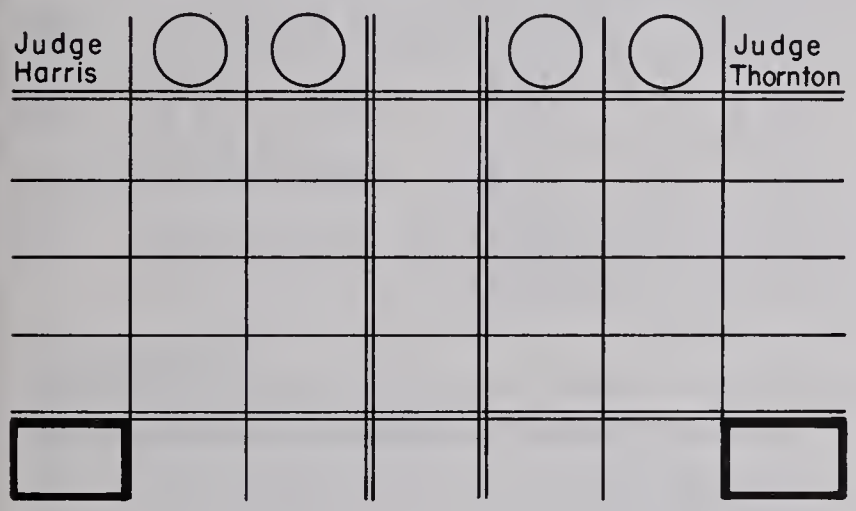
This is an abstraction of the same pattern we see in the "fewest cartons" example. The *Math Workshop* introduces the important idea involved here in the most concrete situations possible. They abound in the study of systems of measurement that

do not follow the pattern of our decimal system. (In fact, our money system is about the only system of measurement we express in decimal notation.)

As we learn to play Sue Marson's games, we will ask the question whether the results follow a pattern that we recognize in the world of events. In this way, we know in each development whether or not the children are grasping the fundamental mathematical idea involved. And we urge teachers to let the idea develop *without verbalization*. Our main concern is the idea and the ability to use it! We have no interest in labeling the ideas for a long time to come. From time to time, as this development unfolds in *Work Text* pages 18-33, you may want to refer to these introductory remarks.

Notes to accompany *Work Text* pages 18-19

On the board, draw the scoreboard or playing board as shown in the first example on page 18—but with no blanks filled in (only the names of the players):

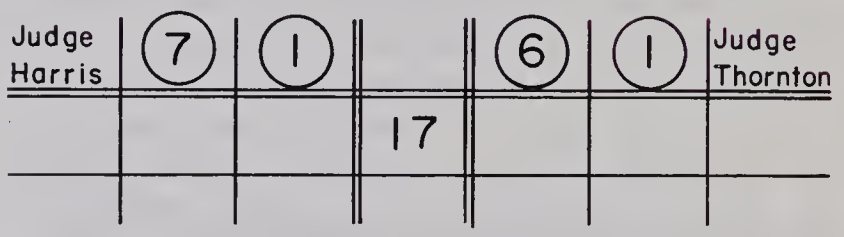


Refer to the *Work Text*. "The finished game is shown. But let's read the rules one at a time and see if we can reconstruct what happened in the game."

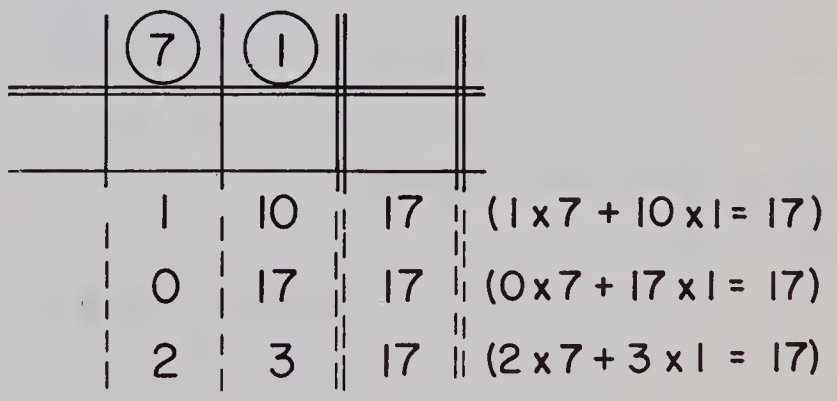
Rule I—Each player writes 1 in the right hand circle on his side of the board. (Fill it in.)

Rule II—Each player picks any number for the other circle. Harris chooses 7, and Thornton chooses 6. (Fill in those circles.)

Rule III—Judge Harris begins the play by writing any number he chooses in the top box in the center column. He chose 17. Now the game board looks like this:



Rule IV—Both players must indicate this amount in terms of multiples of the numbers written in circles on his side of the board. Judge Harris has several ways to do this. Explore them.



"Are these the only choices he has?" (Yes.)

"Which should he choose as the best play?" To make a wise choice, he needs to know Rule V and Rule VII.

In the midst of the discussion of Alec's machine, Mr. Wilson looked up, his eyes wide with a new thought.

"I wonder what kind of project Alec's little sister, Sue, entered in the school fair," he said.

He hurried off, and soon returned with a set of large cards. A neatly printed title was on the top card:

ARITHMETIC GAMES

by Sue Marson

Mr. Wilson and the judges studied Sue's first game. The directions were quite clear.

"Let's play this game," Judge Thornton suggested, and Judge Harris readily agreed.

When they had finished, the game card looked like this:

Judge Harris	Multiply by (7)	(1)		Multiply by (6)	(1)	Judge Thornton
5	2	3	17	2	5	7
4	4	0	28	4	4	8
6	4	2	30	5	0	5
2	2	0	14	2	2	4
17	12	5	89	13	11	24

"Judge Harris won," Mr. Wilson said, "because his final score of 17 is lower than Judge Thornton's 24."

"That bottom check-row was certainly a good idea on Sue's part," Judge Harris remarked. "It's something like a Cross Number Puzzle."

Look at the completed game. How many of Sue's rules can you guess from seeing these results?

RULE I. Each player writes 1 in the circle at the top of the green column on his side of the board.

RULE II. In the circle at the top of the blue column, each player writes any number he chooses. (Judge Harris selected 7 and Judge Thornton chose 6.) These are the multipliers.

RULE III. The first player (Judge Harris) writes any number he wants in the top box in the center column. (Judge Harris chose 17.)

RULE IV. Both players write in the blue and green columns so that the sum of the products of those numbers and the multipliers taken in pairs is the same as the number in the top box of the center column.
(Judge Harris thought:
$2 \times (7) + 3 \times (1) = 14 + 3 = 17$)
(Judge Thornton thought:
$2 \times (6) + 5 \times (1) = 12 + 5 = 17$)

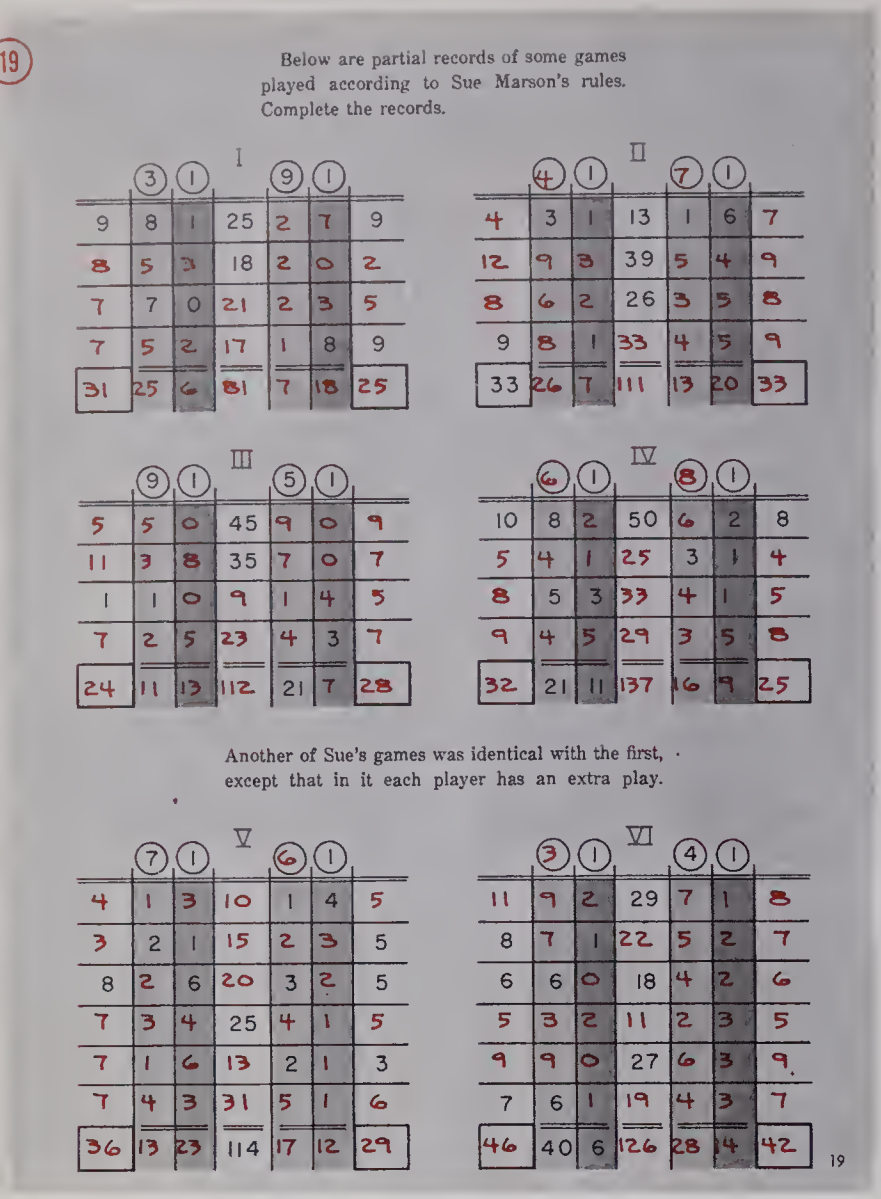
RULE V. Add the last two numbers written in the adjoining space.
$5 = 2 + 3$ and $2 + 5 = 7$

RULE VI. The second player writes any number he chooses in the second box in the center column. (Judge Thornton wrote 28.)

RULE VII. The player with the lowest total in the outside column wins.

Here is a sample game for you to complete:

1st player	(8)	(1)		(5)	(1)	2nd player
4	1	3	11	2	1	3
4	3	1	25	5	0	5
2	1	[illegible]	9	1	4	5
9	3	6	30	6	0	6
19	8	11	75	14	5	19


19
Below are partial records of some games played according to Sue Marson's rules. Complete the records.
I
II
III
IV
Another of Sue's games was identical with the first, except that in it each player has an extra play.
V
VI
19

Rule V—Add the combination of numbers selected and write it in the outside column. If Judge Harris selects 1 and 10, then he writes 11 in the outside column. If he selects 0 and 17, he writes 17 in the outside column. If he selects 2 and 3, he writes 5 in the outside column. So, he must decide whether he wants to write 11, 17 or 5 in that outside box. Rule VII gives him the basis for his choice.

Rule VII—In the end, the player with the lowest scores in the outside columns is the winner, So, Judge Harris, of course, chose 2 and 3.

Next, Judge Thornton must go through a similar study. He must select one of several alternatives:

	6	1	
17			

0	17	(0 x 6 + 17 x 1 = 17)
1	11	(1 x 6 + 11 x 1 = 17)
2	5	(2 x 6 + 5 x 1 = 17)

"Has he any other choices?" (No.) "Which will he select?" He knows that $0 + 17 = 17$, $1 + 11 = 12$ and $2 + 5 = 7$. Since he wants the lowest score, he chooses 2 and 5, and writes 7 in the outside column.

"Who won the first round?" Judge Harris, because his score of 5 is lower than Judge Thornton's score of 7.

Now we need the final rule, Rule VI—The second player then chooses the next number to be written in the center column. Judge Thornton selects 28, and the board now looks like this:

	7	1		6	1	
5	2	3	17	2	5	7
			28			

Judge Harris has several alternate plays to make:

	7	1	
			28

Score ↓			
28	0	28	(0 x 7 + 28 x 1 = 28)
22	1	21	(1 x 7 + 21 x 1 = 28)
16	2	14	(2 x 7 + 14 x 1 = 28)
10	3	7	(3 x 7 + 7 x 1 = 28)
4	4	0	(4 x 7 + 0 x 1 = 28)

He chooses 4 and 0, because that only adds 4 points to his score. Judge Thornton sees his problem as:

	6	1	
28			

		Score ↓	
0	28	28	(0 x 6 + 28 x 1 = 28)
1	22	23	(1 x 6 + 22 x 1 = 28)
2	16	18	(2 x 6 + 16 x 1 = 28)
3	10	13	(3 x 6 + 10 x 1 = 28)
4	4	8	(4 x 6 + 4 x 1 = 28)

Certainly he will choose 4 and 4, because that adds only 8 points to his score.

That's another round for Judge Harris—only 4 points added to his score, but 8 to Judge Thornton's score. Total score for first 2 rounds: Judge Harris: $5 + 4 = 9$; Judge Thornton: $7 + 8 = 15$.

Follow each of the next two rounds in this first game—Judge Harris' second choice of a number for the center column is 30 and Judge Thornton's second choice is 14. At this point, the bottom line shows no record. Add up the 4 scores of Judge Harris and write the total in the lower left-hand corner. Write Judge Thornton's total score in the lower right-hand corner. This line shows:

17						24

Judge Harris wins!

"Now, let's check the results. Add the numbers in the other columns." The bottom line now reads:

17	12	5	89	13	11	24

"On Judge Harris' side, we notice that $12 + 5 = 17$; and on Judge Thornton's side, we notice that $13 + 11 = 24$. Do you suppose this will always happen?" Yes!

"Next, is it true that:

$12 \times 7 + 5 \times 1 = 89$ (Harris) $84 + 5 = 89$
$13 \times 6 + 11 \times 1 = 89$ (Thornton) $78 + 11 = 89$

Double check!

Other examples on pages 18 and 19

The second example on page 18 gives the first 2 plays—and might well be analyzed as each play in the first game was discussed. (The only items missing are the scores for each player in this round.)

The record of the third play is incomplete. The number in the center column is 9. The second player decides to begin with 1 under the circled *5*—$1 \times 5 = 5$ and must then write 4 under the circled *1* for a score of 5 . . . $1 + 4 = 5$. The other player will write 1 and 1 . . . $1 \times 8 + 1 \times 1 = 8 + 1 = 9$ . . . for a low score of 2.

In the last line, the number selected for the center column is not shown. "How can we find out what has been erased?" The second player recorded it as $6 \times 5 + 0 \times 1$, so the fourth choice in the center column must have been 30 . . . etc. Now this game can be completed as the first. It ends in a tie—19 and 19.

On *Work Text* page 19, the jobs of reconstruction become more difficult. Some of the examples seem like puzzles, but whenever all of the numbers that were originally written in the center column have been found, the rest is simply computation.

In example I, the key in the third row is the $7 \times 3 + 0 \times 1 = 21$. In the fourth row, the second player's total score is 9. Since 8 is given in the right column, 1 must have been written in the other; and $1 \times 9 + 8 \times 1 = 17$.

In example II, the fourth line presents a similar problem. If the total score on the left is 9, and one entry is 1, the other must have been 8 since $9 - 1 = 8$ or $9 = 8 + 1$; and $8 \times 4 + 1 \times 1 = 33$.

In III, the key number can be found to be 9. Now when the second player completes the first 3 lines, his entries under the circled *5* are 9, 7 and 1—a total of 17. Since the total in that column is 21 and $21 - 17 = 4$, then his entries on line 4 must be 4 and 3: $4 \times 5 + 3 \times 1 = 20$.

In the first three examples, enough clues of this kind are provided so that, with a little study, the entire game can be reconstructed.

Notes to accompany *Work Text* page 20

The game grows up. Now, instead of choosing any number for the circle to the left of the circled *1*, a spinner is used to select a number. This number becomes a "multiplier" to determine the numbers for the other circles; each circle is a multiple of the one on its right.

In the first game, 2 and 3 are selected by the spinner:

$2 \times 1 = 2$	$3 \times 1 = 3$
$2 \times 2 = 4$	$3 \times 3 = 9$
$2 \times 4 = 8$	$3 \times 9 = 27$
$2 \times 8 = 16$	$3 \times 27 = 81$
$2 \times 16 = 32$	$3 \times 81 = 243$

Now, the game proceeds as the earlier examples. However, the choices of the way to write the numbers chosen for the center column are greater. On the left side of the first game, 11 can be written in many ways:

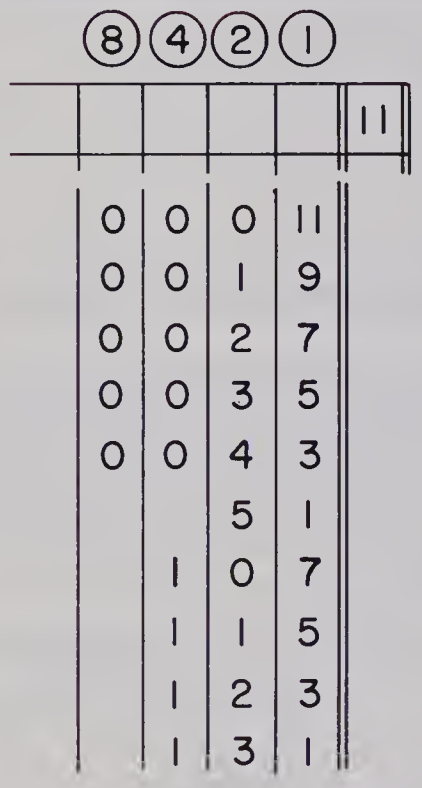
20 Another of Sue Marson's Games.

1. Follow rules given for the last game on page 19.
2. Players take turns filling in any numbers in the center column until it is completed.
3. Each player spins the dial at right.
4. Fill in top row from (1) to the left by multiplying number in previous circle by numbers selected on the dial.

	32	16	8	4	2	1		243	81	27	9	3	1	
3	0	0	1	0	1	1	11	0	0	0	1	0	2	3
3	1	0	0	1	1	0	38	0	0	1	1	0	2	4
3	0	1	1	0	1	0	26	0	0	0	2	2	2	6
6	1	1	1	1	1	1	63	0	0	2	1	0	0	3
15	2	2	3	2	4	2	138	0	0	3	5	2	6	16

	2,401	343	49	7	1		625	125	25	5	1	
7	0	0	1	2	4	67	0	0	2	3	2	7
4	0	2	0	2	0	700	1	0	3	0	0	4
9	0	0	3	0	6	153	0	1	1	0	3	5
4	1	0	2	0	1	2,500	4	0	0	0	0	4
24	1	2	6	4	11	3,420	5	1	6	3	5	20

	6,561	729	81	9	1		10,000	1,000	100	10	1	
11	0	0	0	8	3	75	0	0	0	7	5	12
5	0	0	3	2	0	261	0	0	2	6	1	9
3	1	0	0	1	1	6,571	0	6	5	7	1	19
2	2	0	0	0	0	13,122	1	3	1	2	2	9
21	3	0	3	11	4	20,029	1	9	8	22	9	49



etc.

However, the best score results from:

$$1\times8+0\times4+1\times2+1\times1=8+2+1=11$$

A rather interesting pattern ought to be considered. The first player selected 2 as the multiplier and his scoreboard (except the total) uses numerals that are less than 2—1 and 0.

The second player selected 3 as a multiplier and his scoreboard (except the total) uses numerals that are less than 3—2, 1 and 0. In fact, in each case, no scoreboard shows any numeral (except in totals) as large as the multiplier selected by that player. This is significant.

Finally, the second player in that last game soon found a notable development. His side of the scoreboard closely resembles the numbers chosen in the center column:

	10,000	1,000	100	10	1
75	0	0	0	7	5
261	0	0	2	6	1
6,571	0	6	5	7	1
13,122	1	3	1	2	2

"Why did this happen?"

In Sue Marson's games, the numbers written in the circles give a "value" to that "place." And in the chart above, the value in each place is the same pattern that exists in our decimal system of numeration —moving to the left and starting with units, each place has a value of 10 times its predecessor. Also, to write in our demical notation, we use no numeral in any place greater than 10-1 . . . a pattern in all examples on this page. Thus, insight into our decimal system of notation deepens.

Notes to accompany *Work Text* page 21

The first example on this page shows that 42 could be written as $0 \times 25 + 7 \times 5 + 7 \times 1 = 35 + 7 = 42$. Now, if this example came up on one of the earlier games, no player would play in that way—he would get a score of $0 + 7 + 7$ or 14. He would undoubtedly think $1 \times 25 + 3 \times 5 + 2 \times 1 = 25 + 15 + 2 = 42$—and his score would only be $1 + 3 + 2$ or 6.

This same thinking underlies the game we are playing on *Work Text* page 21. In each example, one way of indicating the key number (42, in the first) is given in whole or in part. Enough is given so there is only one way to complete the top or green-tinted row.

In the blue-tinted row, the problem is to find a way of indicating the same number in such a way that it would be the lowest score in the earlier games.

A second look at this page

When this page is completed, look at the examples from a different point of view. Consider the first example. Suppose Bill has 42¢. As he counts out the coins, he puts all his nickels in one pile. There are only 7. What other coins does he have? He must have 7 pennies. Check: $7 \times 5 + 7 \times 1 = 35 + 7 = 42$.

Now suppose he wants to change this into the "fewest coins" . . . but dimes are excluded. How could he do it? Answer: with 1 quarter, 3 nickels and 2 pennies: $1 \times 25 + 3 \times 5 + 2 \times 1 = 42$.

The similarity between this problem and the first example on page 21 is certainly obvious.

With an eye to the second example, consider this situation: Cookies are packed in packages of 6 each. They also come in a box of 6 of these packages—or 6×6, or 36 cookies. Bill has 5 packages (6 each) and 12 separate cookies. He wants them in the fewest number of full boxes; full packages . . . with the fewest left over.

He has $5 \times 6 + 12$, or 42 cookies. So, he packs 36 in 1 box, 6 in 1 package, and has none left over.

Now keep an eye on the third example in the top line. The first two grades in Lowell School need 42 glasses of milk. The store has half-pint cartons, quarts and gallons. Now there are 4 half-pints in a quart and 16 half-pints in a gallon. One way they could buy 42 half-pints would be in 2 gallons, 1 quart and 6 half-pints—because $2 \times 16 + 1 \times 4 + 6 \times 1 = 32 + 4 + 6 = 42$.

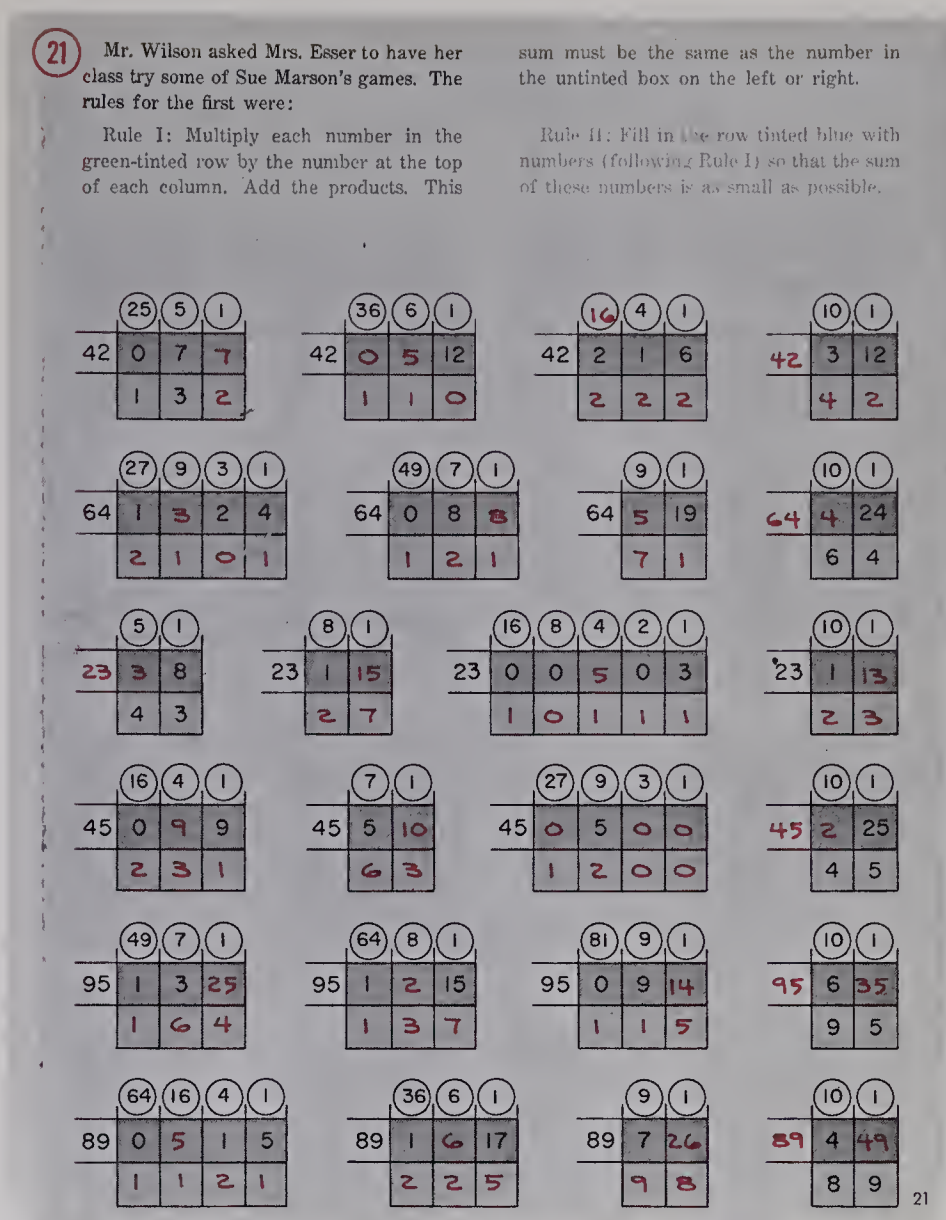
But how can they make the purchase in the fewest such cartons? Answer: 2 gallons, 2 quarts and 2 half-pints—$2 \times 16 + 2 \times 4 + 2 \times 1 = 32 + 8 + 2 = 42$ . . . and that is only 3 cartons.

Glance at the fourth example in the top line. Paul had 3 dimes and 12 pennies. He changed that into the fewest coins that were either dimes or pennies. He had 42¢ . . . and changed it into 4 dimes and 2 pennies—6 coins.

On a certain star, there are 7 days in a week and 7 weeks or 49 days in a "moonth."

There are 9 players on a baseball team. There are 5 players on a basketball team, etc.

1 half-pint = 1 half-pint
2 half-pints = 1 pint
4 half-pints = 1 quart
8 half-pints = 1 half-gallon
16 half-pints = 1 gallon


21
Mr. Wilson asked Mrs. Esser to have her class try some of Sue Marson's games. The rules for the first were:
Rule I: Multiply each number in the green-tinted row by the number at the top of each column. Add the products. This sum must be the same as the number in the untinted box on the left or right.
Rule II: Fill in the row tinted blue with numbers (following Rule I) so that the sum of these numbers is as small as possible.

There are 3 legs on a 3-legged stool.

There are 9 legs in a box of three 3-legged stools.

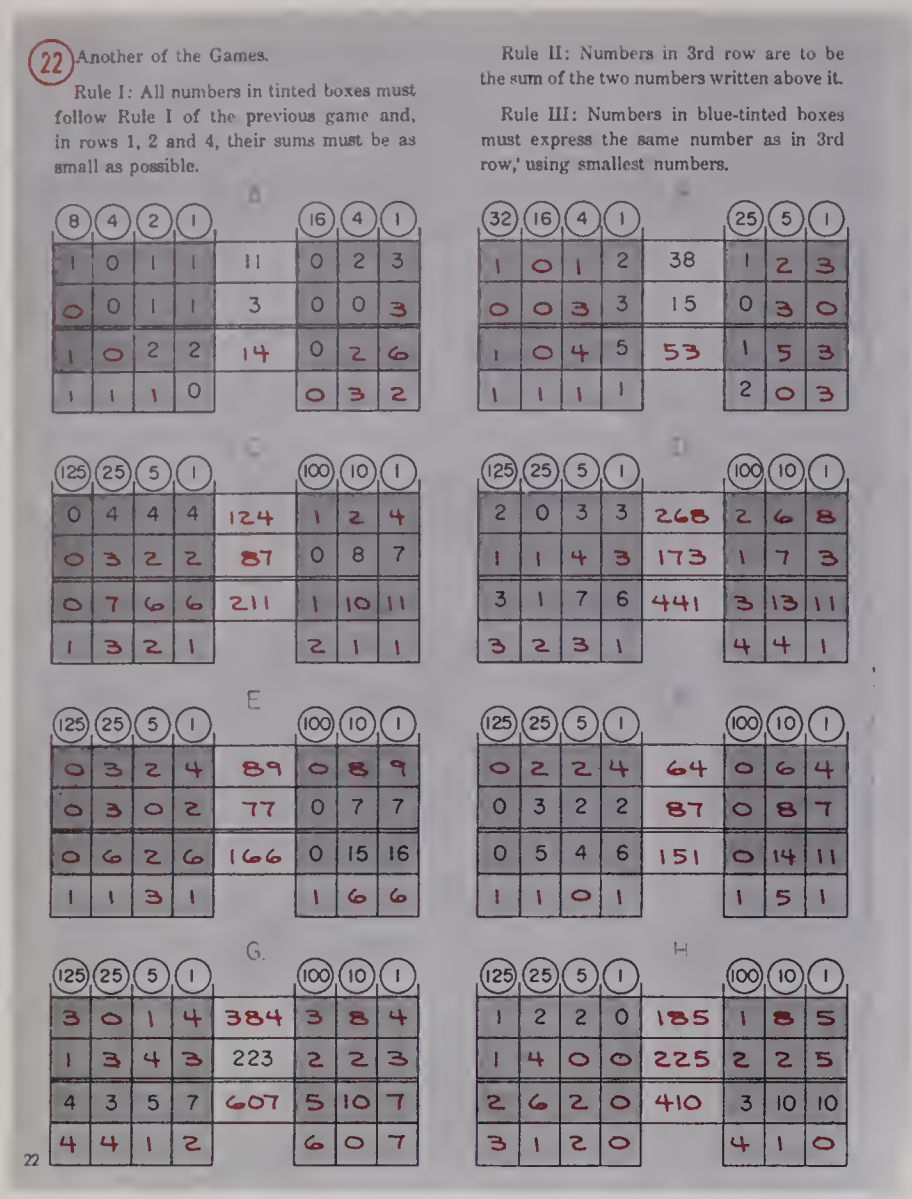
There are 27 legs in a carton of 3 boxes each containing three 3-legged stools.

Notes to accompany *Work Text* pages 22-24

On pages 22, 23 and 24, the idea introduced on page 21 is combined with earlier versions of Sue's games. Each of two players suggest a key number for the center column. Then each indicates that number so his score will be as low as possible.

Now, add the numbers in each column and write the results in the third row. Usually, the total in the center column could be written with a lower score than the third row would indicate. So, in the blue area, write it so the score is the least.

In example A, the first player's score at row three is $1 + 0 + 2 + 2$ or 5. The total in the center column is 14, which the first player could write as $1 \times 8 + 1 \times 4 + 1 \times 2 + 0 \times 1 = 8 + 4 + 2 = 14$ . . . and his "score" is only 3.



22 Another of the Games.

Rule I: All numbers in tinted boxes must follow Rule I of the previous game and, in rows 1, 2 and 4, their sums must be as small as possible.

Rule II: Numbers in 3rd row are to be the sum of the two numbers written above it.

Rule III: Numbers in blue-tinted boxes must express the same number as in 3rd row, using smallest numbers.

A

(8)	(4)	(2)	(1)		(16)	(4)	(1)
1	0	1	1	11	0	2	3
0	0	1	1	3	0	0	3
1	0	2	2	14	0	2	6
1	1	1	0		0	3	2

B

(32)	(16)	(4)	(1)		(25)	(5)	(1)
1	0	1	2	38	1	2	3
0	0	3	3	15	0	3	0
1	0	4	5	53	1	5	3
1	1	1	1		2	0	3

C

(125)	(25)	(5)	(1)		(100)	(10)	(1)
0	4	4	4	124	1	2	4
0	3	2	2	87	0	8	7
0	7	6	6	211	1	10	11
1	3	2	1		2	1	1

D

(125)	(25)	(5)	(1)		(100)	(10)	(1)
2	0	3	3	268	2	6	8
1	1	4	3	173	1	7	3
3	1	7	6	441	3	13	11
3	2	3	1		4	4	1

E

(125)	(25)	(5)	(1)		(100)	(10)	(1)
0	3	2	4	89	0	8	9
0	3	0	2	77	0	7	7
0	6	2	6	166	0	15	16
1	1	3	1		1	6	6

F

(125)	(25)	(5)	(1)		(100)	(10)	(1)
0	2	2	4	64	0	6	4
0	3	2	2	87	0	8	7
0	5	4	6	151	0	14	11
1	1	0	1		1	5	1

G.

(125)	(25)	(5)	(1)		(100)	(10)	(1)
3	0	1	4	384	3	8	4
1	3	4	3	223	2	2	3
4	3	5	7	607	5	10	7
4	4	1	2		6	0	7

H

(125)	(25)	(5)	(1)		(100)	(10)	(1)
1	2	2	0	185	1	8	5
1	4	0	0	225	2	2	5
2	6	2	0	410	3	10	10
3	1	2	0		4	1	0

22

The second player's first total score (line 3) is 8, because he has expressed 14 as $2 \times 4 + 6 \times 1 = 14$. He could improve his score by writing 14 as $3 \times 4 + 2 \times 1 = 14$. His adjusted score is 3 + 2 or 5.

But, the first player wins; his score of 3 is less than the second player's score of 5.

In the activities on *Work Text* pages 22, 23 and 24, the bottom row—blue in each case—calls for this same kind of conversion, indicating the total with digits so the sum of those digits is as low as possible.

Each page starts out with straight-forward examples, but as the activity proceeds, fewer and fewer clues are given as a basis for reconstructing the game as it was played.

After the first two games on page 22, all the rest of the games on that page can be thought of in the same way. One player is working with pennies, nickels, quarters and buck-and-a-quarter pieces (or boxes of 5 quarters each); the other is working with pennies, dimes and silver dollars. The green line expresses the total amount in "fewest coins."

Work Text page 23 is a study in depth of the "pennies, nickels, quarters" version. Only numerals less than 5 are permitted.

Each time another game is studied, encourage children to play it on their own. If necessary, duplicate some blanks for them to use. Let them select their own numbers for the circles, as in the earliest version. They will soon discover strategies of play—and each successful strategy leads them to deeper understanding of our systems of numeration.

Now, if you want, you can explain that mathematicians have a way of putting this idea in a different form. When place value is determined by starting on the right with 1 and then assigning a value to each place by multiplying the previous place by 5, the mathematicians refer to this as working "in the base 5," and they write "FIVE" to tell everyone not to forget that the places from right to left have a value of 1, 5, 25, 125, etc.

They would write the top row of examples this way:

	3	12	14
	FIVE	FIVE	FIVE
	1	3	14
+	FIVE	FIVE	FIVE
	4	20	33
	FIVE	FIVE	FIVE
	4	33	23
	FIVE	FIVE	FIVE
	11	3	14
+	FIVE	FIVE	FIVE
	20	41	42
	FIVE	FIVE	FIVE

Or the mathematicians might say very clearly at the top of a series of examples what he is doing—and then he doesn't have to write that "FIVE" all the time.

"In this series of examples, I am working in Base FIVE."

10	2	13	3	14	4
23	3	3	14	4	20
+11	+4	+13	+3	+14	+4
44	14	34	30	42	33

You may want the children to work these examples on the board. They are the second row on *Work Text* page 23. If the children get stuck, have them think in "pennies and nickels" and "fewest coins."

Some may want to explore this idea further. If so, warn them that they ought always to write in bold letters at the top of the page that "place value" has a special meaning on that page . . . unless they want to work in a "secret code"—because, unless otherwise noted, all arithmetic is done in base 10, with place values from right to left—1, 10, 100, 1000, etc.

A change

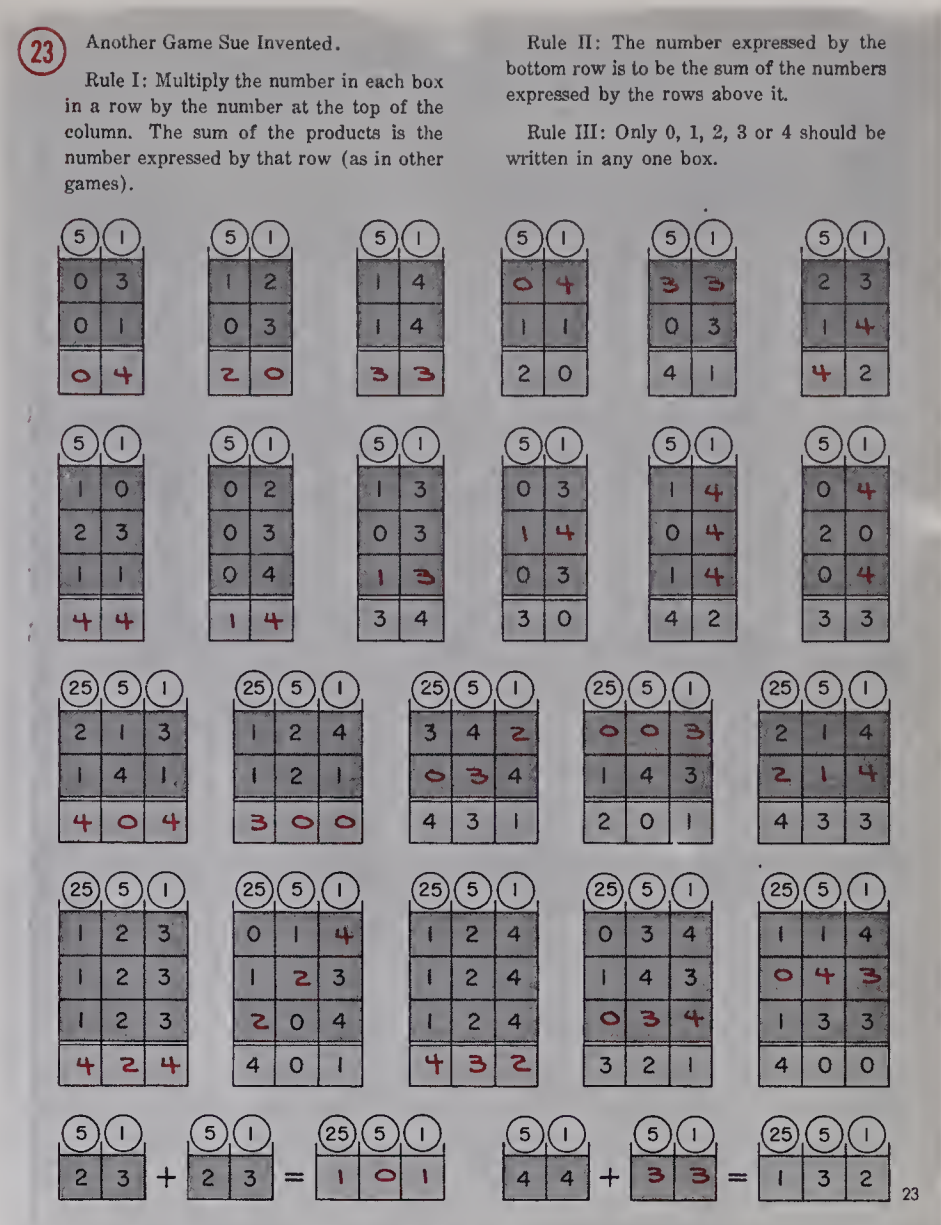
On page 24, no numeral can be written in any column of any example that is as large as the number written in the blue-tinted area. There is only one way to complete or reconstruct any of the examples. In several, considerable ingenuity is required to find that way.

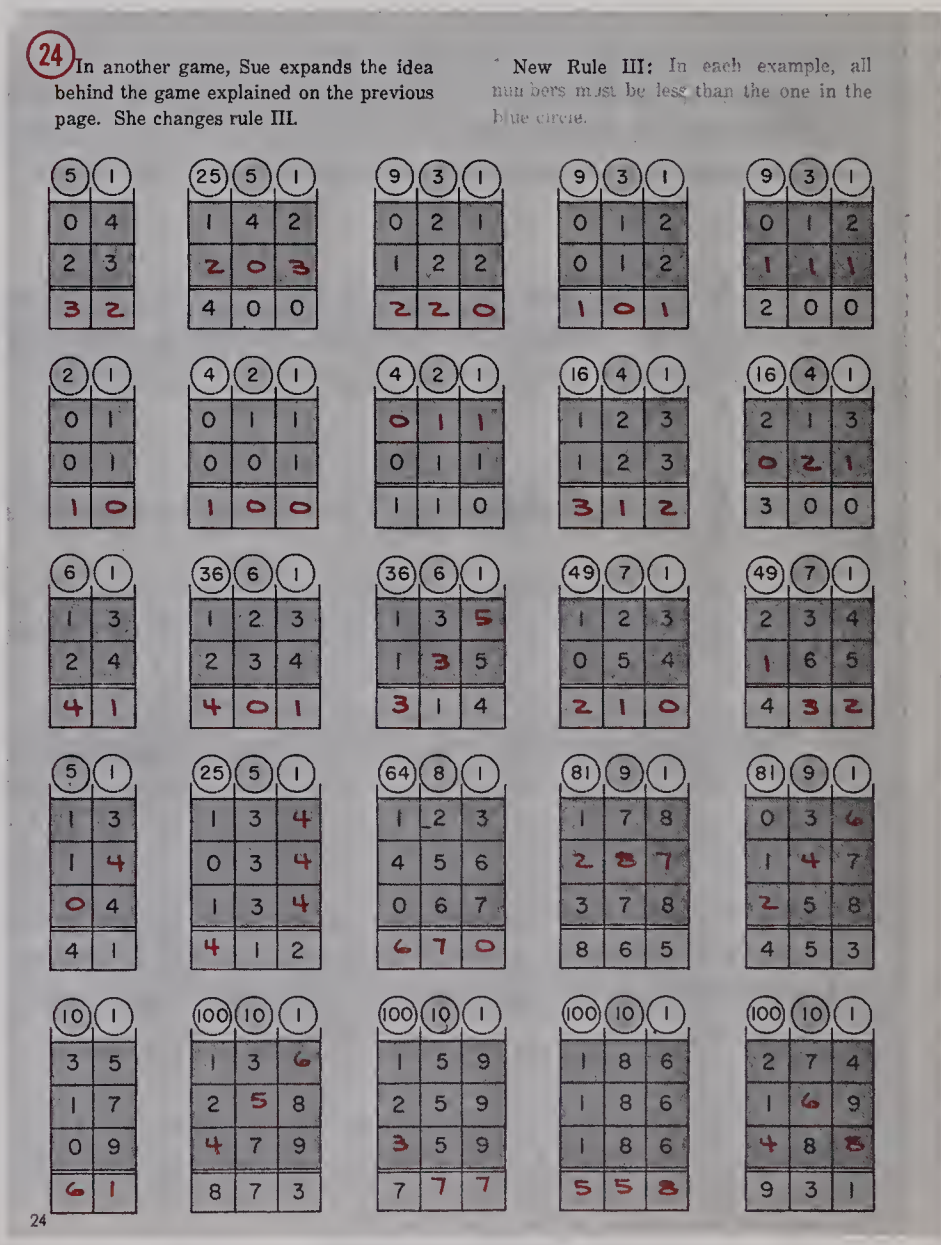
If you want to explore the idea of numbers to other bases further, show the children that certain of the examples on page 24 would look like this:

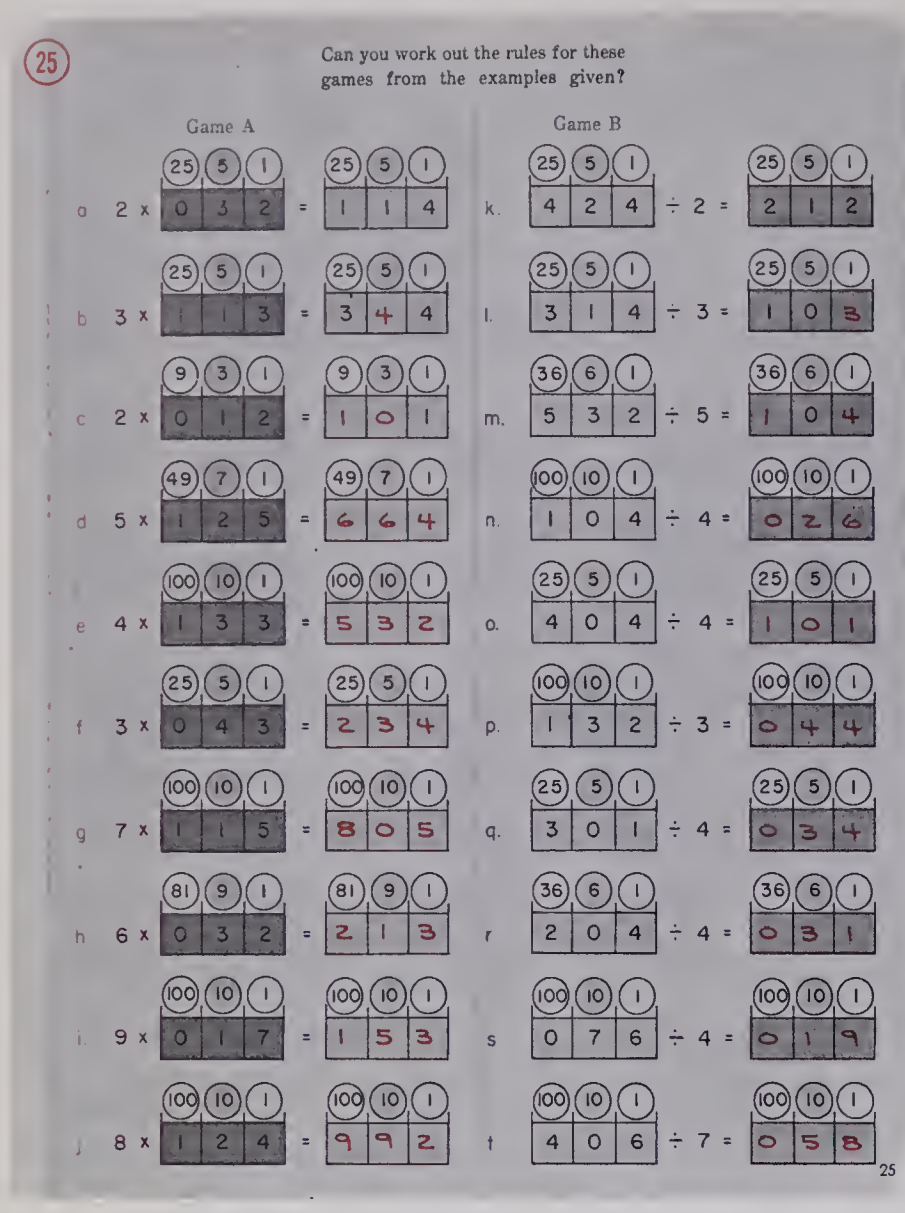
21	123	13
THREE	FOUR	SIX
122	123	24
+ THREE	FOUR	SIX
220	312	41
THREE	FOUR	SIX
123	123	178
SEVEN	EIGHT	NINE
54	456	287
+ SEVEN	EIGHT	NINE
210	67	378
SEVEN	EIGHT	NINE
	670	865
	EIGHT	NINE
35	136	274
TEN	TEN	TEN
17	258	169
TEN	TEN	TEN
9	479	488
+ TEN	TEN	TEN
61	873	931
TEN	TEN	TEN

"Were the indications of the base required in these last 3 examples?" No! If no base is indicated, it is to be assumed we are working in base 10—where each place has a value of 10 times the one on its right.

Perhaps this extension of Sue's games into an explanation of what mathematicians mean by other bases is so strange and unfamiliar that some teachers will feel uncomfortable. Those teachers who do should either avoid it completely, or simply indicate the idea.


23
Another Game Sue Invented.
Rule I: Multiply the number in each box in a row by the number at the top of the column. The sum of the products is the number expressed by that row (as in other games).
Rule II: The number expressed by the bottom row is to be the sum of the numbers expressed by the rows above it.
Rule III: Only 0, 1, 2, 3 or 4 should be written in any one box.
23


24
In another game, Sue expands the idea behind the game explained on the previous page. She changes rule III.
New Rule III: In each example, all numbers must be less than the one in the blue circle.
24



Our main concern is with the idea of different systems of numerations as it has important applications in handling different systems of measurement. This idea is rooted deeply in Sue's games and the analogies we draw between patterns in the games and patterns in the world of events.

Children will meet "other bases" again. If some are already excited by the idea, let them find out for themselves how they might subtract in "different bases."

Notes to accompany *Work Text* page 25

The rule is still in effect that all numerals used may be less than the one indicated in the blue-tinted area of the circle in each example.

Consider example *a* as expressing the idea that Bill had 3 nickels and 2 pennies. Next day, he earned the same amount, so he had twice as much as 3 nickels and 2 pennies. He changed it into the fewest number of coins (including only pennies, nickels and quarters). Six nickels and 4 pennies is the same amount of money as 1 quarter, 1 nickel and 4 pennies—34 cents.

Think of example *c* this way: Tennis balls are packed 3 in a can and 3 cans in a box (or 9 tennis balls in a box). One can and an extra tennis ball were on display. Harry said, "I want twice as many, and in the fewest packages." The clerk gave him 1 box and 1 loose tennis ball.

Again, invent the planet on which a week is 7 days and a "moonth" is 7 weeks or 49 days. If it took 1 moonth, 2 weeks and 1 day to walk one-fifth of the way around this planet, at the same rate, how long would it take to make a complete journey around the planet? Five times as long.

1 moonth
× 5
5 moonths

2 weeks
× 5
(10 weeks, but 6 is the largest numeral, so we change this to) 1 moonth and 3 weeks

5 days
× 5
(25 days, changed to) 3 weeks and 4 days

5 moonths
1 moonth and 3 weeks
\+ 3 weeks and 4 days
6 moonths and 6 weeks and 4 days

Example *e* is a most familiar pattern. It suggests the idea of working in coins limited to pennies, dimes and silver dollars. Models were reduced in price from $1.50, so Al bought one with 1 dollar, three dimes, and three pennies. If he had bought 4, how could he pay for them with the fewest coins?

1 dollar
× 4
4 dollars

3 dimes
× 4
(12 dimes, but we must change this to) 1 dollar and 2 dimes

3 pennies
× 4
(12 cents, but we must change this to) 1 dime and 2 pennies

4 dollars
1 dollar and 2 dimes
\+ 1 dime and 2 pennies
5 dollars and 3 dimes and 2 pennies

Example *g* has an additional conversion to make. Mary paid one dollar, one dime and 5 pennies for a

book. The clerk's chart showed how much the 7 books, at that price, would cost:

1 dollar
× 7
7 dollars

1 dime
× 7
7 dimes

5
× 7
(35, But we must change this to) 3 dimes and 5 pennies

7 dollars
7 dimes
3 dimes and 5 pennies
(10 dimes, changed to) 1 dollar

so we write:

7 dollars
1 dollar and no dimes and 5 pennies
8 dollars and 0 dimes and 5 pennies

"Do we go through similar steps when we use ordinary arithmetic?"

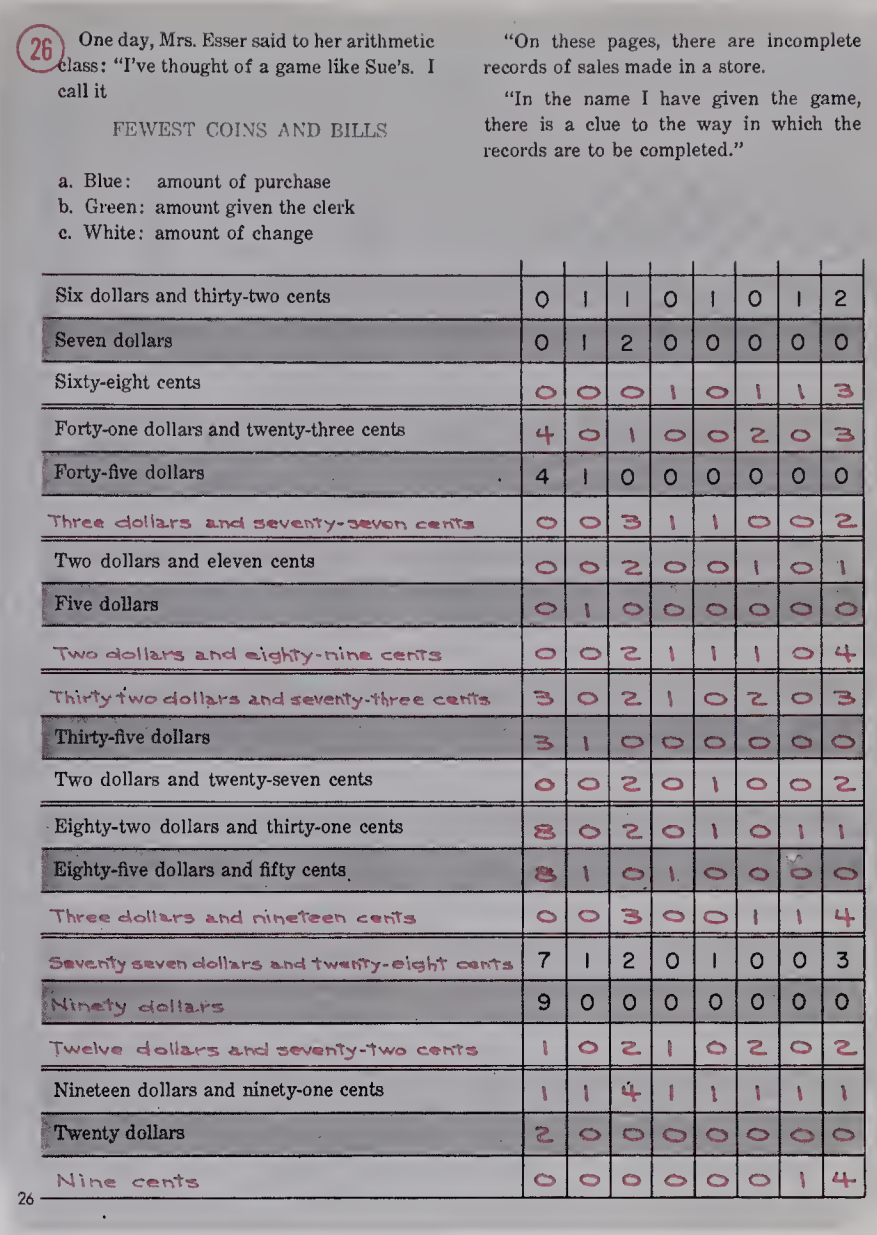
$1.15
× 7
5 (we count 35¢ to 3 dimes and 5¢. We write down the 5¢ and remember "3 dimes")

$1.15
× 7
05 (7 × 1 dime—or 10¢—is 7 dimes or 70¢ plus the 30¢ or 3 dimes "remembered" is 100¢ or 10 dimes or 1 dollar and no dimes—0 dimes and remember 1 dollar)

$1.15
× 7
$8.05 (7 × 1 dollar plus the 1 dollar remembered is 8 dollars. Write it down.)

In similar manner, each of the examples *a* through *j* can be analyzed. Examples *k* through *t* are similar —but involve the idea of division.

"Jean had 4 quarters, 2 nickels and 4 pennies to divide with her sister—using only quarters, nickels and pennies. She gave her sister 2 quarters, 1 nickel and 2 pennies."



26 One day, Mrs. Esser said to her arithmetic class: "I've thought of a game like Sue's. I call it

FEWEST COINS AND BILLS

"On these pages, there are incomplete records of sales made in a store.

"In the name I have given the game, there is a clue to the way in which the records are to be completed."

a. Blue: amount of purchase
b. Green: amount given the clerk
c. White: amount of change

Six dollars and thirty-two cents	0	1	1	0	1	0	1	2
Seven dollars	0	1	2	0	0	0	0	0
Sixty-eight cents	0	0	0	1	0	1	1	3
Forty-one dollars and twenty-three cents	4	0	1	0	0	2	0	3
Forty-five dollars	4	1	0	0	0	0	0	0
Three dollars and seventy-seven cents	0	0	3	1	1	0	0	2
Two dollars and eleven cents	0	0	2	0	0	1	0	1
Five dollars	0	1	0	0	0	0	0	0
Two dollars and eighty-nine cents	0	0	2	1	1	1	0	4
Thirty-two dollars and seventy-three cents	3	0	2	1	0	2	0	3
Thirty-five dollars	3	1	0	0	0	0	0	0
Two dollars and twenty-seven cents	0	0	2	0	1	0	0	2
Eighty-two dollars and thirty-one cents	8	0	2	0	1	0	1	1
Eighty-five dollars and fifty cents	8	1	0	1	0	0	0	0
Three dollars and nineteen cents	0	0	3	0	0	1	1	4
Seventy seven dollars and twenty-eight cents	7	1	2	0	1	0	0	3
Ninety dollars	9	0	0	0	0	0	0	0
Twelve dollars and seventy-two cents	1	0	2	1	0	2	0	2
Nineteen dollars and ninety-one cents	1	1	4	1	1	1	1	1
Twenty dollars	2	0	0	0	0	0	0	0
Nine cents	0	0	0	0	0	0	1	4

26

"Pete spent ⅓ of his 3 quarters, 1 nickel and 4 pennies. He did it by spending 1 quarter and 3 pennies. He had 2 quarters, 1 nickel and 1 penny left."

Notes to accompany *Work Text* pages 26-27

While our money system is based on the decimal system of numeration we have added some coins and bills that interrupt the basic pattern.

On *Work Text*, page 26, we admit those additions —nickels, quarters, half-dollars and $5 bills. (The $2 bill is hard to find in most of the country.) It takes considerable alertness to be sure we are always expressing an amount in the fewest coins and bills.

On *Work Text* page 27, however, we admit *only* those coins and bills that follow the place value in our decimal system as it is applied to money: pennies, dimes (10 pennies), dollars (10 dimes), ten dollar bills (10 dollars), and hundred dollar bills (10 ten-dollar bills). Now, we are on completely familiar ground—and duly aware of the very material advantages in a decimal system without the clutter of items that do not fit the pattern.

27 "Now let's change the rules a bit," Mrs. Esser said.

New Rule: The only coins and bills to be used are pennies, dimes, one dollar bills, ten dollar bills and one hundred dollar bills.

"And, instead of writing the amount in words, we will use shorthand."

	Money Shorthand	One hundred dollar bills	Ten dollar bills	One dollar bills	Dimes	Pennies
I	$.93	0	0	0	9	3
	$5.00	0	0	5	0	0
	$4.07	0	0	4	0	7
II	$57.63	0	5	7	6	3
	$60.00	0	6	0	0	0
	$2.37	0	0	2	3	7
III	$125.25	1	2	5	2	5
	$130.00	1	3	0	0	0
	$4.75	0	0	4	7	5
IV	$21.37	0	2	1	3	7
	$30.00	0	3	0	0	0
	$8.63	0	0	8	6	3
V	$190.87	1	9	0	8	7
	$200.00	2	0	0	0	0
	$9.13	0	0	9	1	3
VI	$76.17	0	7	6	1	7
	$200.50	2	0	0	5	0
	$124.33	1	2	4	3	3
VII	$18.84	0	1	8	8	4
	$101.00	1	0	1	0	0
	$82.16	0	8	2	1	6
VIII	$191.05	1	9	1	0	5
	$200.00	2	0	0	0	0
	$8.95	0	0	8	9	5
IX	$13.84	0	1	3	8	4
	$20.90	0	2	0	9	0
	$7.06	0	0	7	0	6
X	$371.36	3	7	1	3	6
	$580.00	5	8	0	0	0
	$208.64	2	0	8	6	4
XI	$681.29	6	8	1	2	9
	$701.51	7	0	1	5	1
	$20.22	0	2	0	2	2
XII	$66.69	0	6	6	6	9
	$96.00	0	9	6	0	0
	$29.31	0	2	9	3	1
XIII	$755.18	7	5	5	1	8
	$800.50	8	0	0	5	0
	$45.32	0	4	5	3	2
XIV	$836.54	8	3	6	5	4
	$901.00	9	0	1	0	0
	$64.46	0	6	4	4	6
XV	$407.83	4	0	7	8	3
	$510.00	5	1	0	0	0
	$102.17	1	0	2	1	7

27

The examples on page 26 are similar to playing one of Sue Marson's games that had the following heading (see the first example):

	(1,000)	(500)	(100)	(50)	(25)	(10)	(5)	(1)	
line A	0	1	1	0	1	0	1	2	632
line B	0	1	2	0	0	0	0	0	700
line C	○	○	○	!	○	!	!	3	

. . . and the directions would be: "Subtract the number indicated by line A from the number indicated in line B. Record the results in line C . . . and keep the score for each line as low as possible."

The answer with the lowest score would be "1 half-dollar, 1 dime, 1 nickel and 3 pennies—68 cents."

Enough information is given in each example on both pages so that there is only one way to reconstruct the experience.

As children are working their way through to a sensible conclusion in these and other games by Sue Marson, they are continually confronted with the concept of place value and the system of numeration that depends on it. Time and again they must refer to the particular scheme of assigning "value" to a "place" in each expression. They encounter obstacles . . . and must think their way through to a conclusion.

The system keeps changing. There is no economy in memorizing detailed rules because, before they are worked out, another example looms ahead that would require its own rules. To move ahead, the only course of action is to *understand* what's going on.

Children who have been drilled in place value as it works out in the decimal system, have learned rules they can follow . . . but they have no real understanding of the meaning of place value in its general sense. "Place value" to them means—"units, tens, hundreds, thousands, etc.," and, later, going to the right of the decimal point, "tenths, hundredths, thousandths." They do not understand that this is only a special case of a general concept. Thus, they do not understand that, in a very true sense, all our systems of measurement are based on the same place value concept.

Because most of our systems of measurement depart widely from the dictates of logic, they must learn to add measurements of distance in this way:

$$\begin{array}{r} 3 \text{ yds. and } 2 \text{ ft. and } 7 \text{ inches and } \frac{1}{2} \text{ inch} \\ +\ 1 \text{ yd. and } 5 \text{ ft. and } 9 \text{ inches and } \frac{3}{4} \text{ inch} \\ \hline 5 \text{ yds. and } 1 \text{ ft. and } 5 \text{ inches and } \frac{1}{4} \text{ inch} \end{array}$$

It is in such a context as this that "carrying" and (if we called for subtraction) "borrowing" find their true generality.

Later, we will consider a very common system of measurement in which place value depends on a "multiplier" of 60 in part:

$$\begin{array}{rrr} 27^\circ & 19' & 39'' \\ +\ 12^\circ & 14' & 41'' \\ \hline 40^\circ & 14' & 20'' \end{array}$$

This occurs when we consider the measure of angles. We read the problem as, "Find the sum of 27 degrees, 19 minutes and 39 seconds, and 12 degrees, 54 minutes and 41 seconds." The answer: 41 degrees, 14 minutes and 20 seconds.

Children who have become adept at playing Sue Marson's games will understand this important

scheme of handling addition and subtraction of measurements, and multiplication and division of measurements by a number.

The purpose of Sue's games is to help children grasp a kind of pattern that is fundamental to the idea of numeration. Because our primary concern is with the idea and not language, the *Work Text* does not encourage verbalization at this point.

Notes to accompany *Work Text* pages 28-29

Pages 28 and 29 focus attention on our system of measuring time. The pattern of this system is a special one—with no consistent basis. Some of the units for measuring time are:

60 seconds = 1 minute
60 minutes = 1 hour
24 hours = 1 day
7 days = 1 week
____ weeks = 1 month
12 months = 1 year
100 years = 1 century

This is a clumsy system—but we must understand it. If we add 365 days = 1 year, except on "leap year" (when 366 days = 1 year), we complicate it further. And then we go on to make exceptions about leap years which *usually* occur in years assigned a number that is divisible by four.

Nevertheless, this is one of the most important of our systems of measurements. We need to understand clearly the essential relationships of its units.

A blank is left in the chart because the number of weeks in a month varies from 4 to $4\frac{1}{7}$ to $4\frac{2}{7}$ to $4\frac{3}{7}$, depending on the month and year. (February has $4\frac{1}{7}$ weeks in leap year.) To avoid this troublesome factor, the examples are limited to "weeks" as the largest unit, and "11 weeks" is not translated into any other unit of measure for time.

The game on *Work Text* page 28 deals with conversion of time so that the score of units of time is "lowest."

A. "8 days" can be written with a "lower score" as "1 week and 1 day"—the game of "fewest units," it might be called.

28

"This kind of game," Mrs. Esser explained, "needn't be limited to amounts of money. Here are rules and examples involving ways to express the same amounts of time.

Rule I: Blue and green rows between double-black lines express the same amount of time.

Rule II: Blue row always uses smallest possible total of units."

	Weeks	Days
A	0	8
	1	1
B	3	21
	6	0
C	17	17
	19	3
D	6	6
	6	6

	Days	Hours
E	1	24
	2	0
F	0	49
	2	1
G	10	75
	13	3
H	7	33
	8	9

	Hours	Min.
I	2	80
	3	20
J	0	180
	3	0
K	5	137
	7	17
L	0	617
	10	17

	Min.	Sec.
M	0	185
	3	5
N	13	47
	13	47
O	2	637
	12	37
P	9	330
	14	30

	Min.	Sec.
Q	1	300
	6	0
R	3	195
	6	15
S	2	58
	2	58
T	12	158
	14	38

	Hours	Min.	Sec.
a	0	65	60
	1	6	0
b	10	240	98
	14	1	38
c	7	52	483
	8	0	3

	Weeks	Days	Hours
d	1	7	26
	2	1	2
e	19	3	100
	20	0	4
f	2	70	72
	12	3	0

	Days	Hours	Min.
g	5	11	100
	5	12	40
h	1	48	600
	3	10	0
i	12	39	39
	13	15	39

	Hours	Min.	Sec.
j	11	120	87
	13	1	27
k	9	90	99
	10	31	39
l	7	50	665
	8	1	5

	Weeks	Days	Hours
m	3	15	50
	5	3	2
n	7	7	70
	8	2	22

	Days	Hours	Min.	Sec.
o	0	30	59	125
	1	7	1	5
p	4	26	69	87
	5	3	10	27

	Weeks	Days	Hours	Min.	Sec.
q	1	11	24	90	125
	2	5	1	32	5
r	0	9	19	600	128
	1	3	5	2	8

28

B. "3 weeks and 21 days" can be written in "fewest weeks" or "lowest score" as "6 weeks—a score of 3 + 21 is thus reduced to a score of 6.

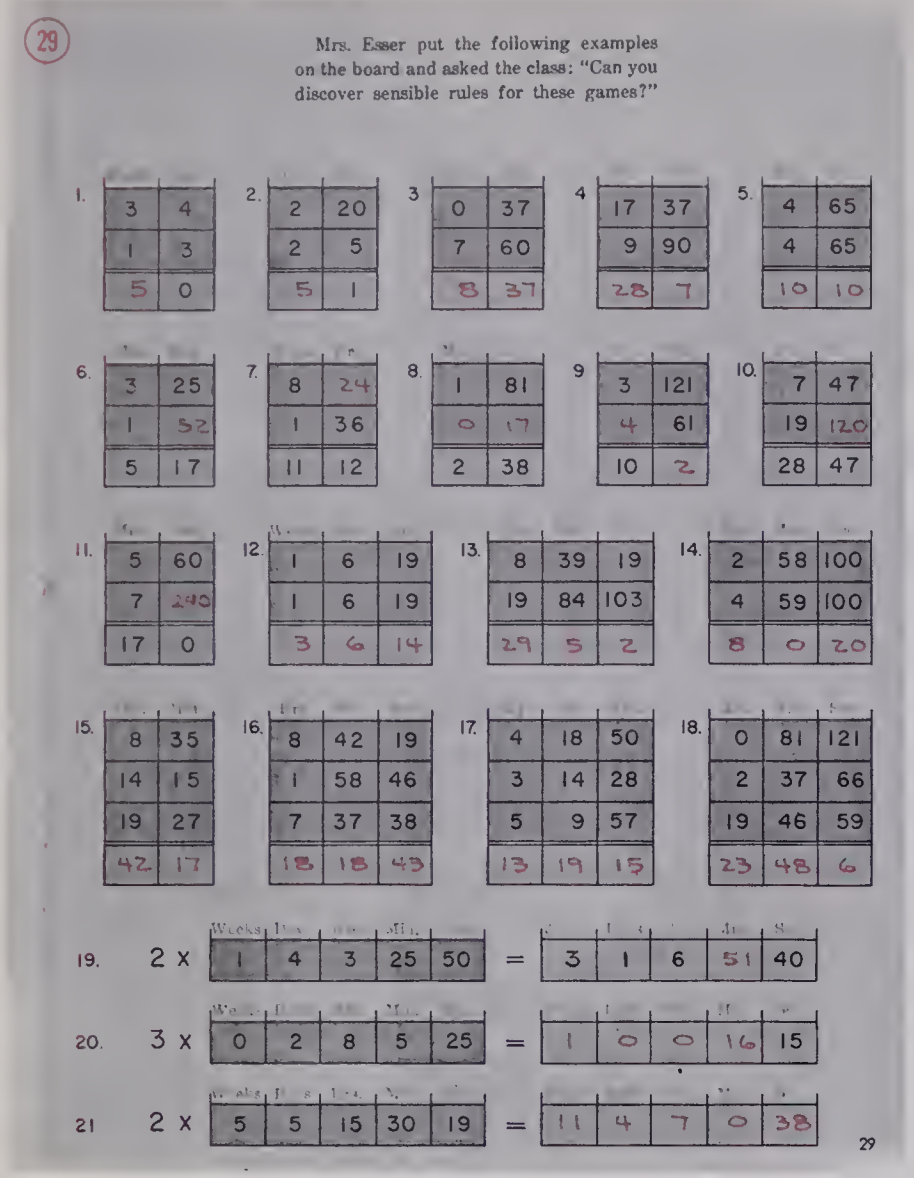
Because of the many games already played, *lowest score* has a specific meaning that does not have to be defined. Any attempt to define the idea would degenerate into a semantic exercise and lead away from mathematics. "Lowest score" can, at this point, be accepted as a name for a recurrent pattern in Sue Marson's games. It needs no further explanation.

C. "17 weeks and 17 days" could be written with a lowest score as "19 weeks and 2 days."

D. "6 weeks and 6 days" cannot be written with a lowest score in any other way.

On this page, each group of two lines is an isolated example. A length of time is indicated in a green-tinted row; that length of time is to be indicated in the blue-tinted row below in terms of lowest score.

Work Text page 29 takes a different direction. The first example asks, "How can 3 weeks and 4 days be added to 1 week and 3 days—and the sum recorded with the 'lowest score?' " Answer: 5 weeks.



On page 29, some of the examples require considerable ingenuity, but enough clues are given in all examples on both pages to force only one conclusion. Example 6 says, in effect, that the sum of two periods of time is 5 hours and 17 minutes. One period was 3 hours and 25 minutes, and the other was something more than 1 hour. How much more than an hour was it?

At the bottom of page 29, there are examples that consider multiplying a length of time by a number. Consider example 19. It might be concerned with a trip to the coast which took 1 week, 4 days, 3 hours, 25 minutes and 50 seconds. "If the return trip were made in the same time, how much time would the round-trip require?"

Examples 19, 20 and 21 involve multiplying a length of time by a number. You might ask for examples in which "dividing a length of time by a number" would make sense. An illustration might be: "Three orbits around the earth by a space capsule took 4 hours and 15 minutes; how long did it take for 1 orbit if the capsule traveled at a uniform speed?" Answer: 1 hour and 25 minutes.

Urge children to make up their own stories or events in which lengths of time are added or subtracted, or in which lengths of time are multiplied or divided by a number.

Notes to accompany *Work Text* pages 30-33

Each chart on pages 30 through 33 is composed exclusively of examples in which the student multiplies a measurement by a number—2 × 7.13 or 6 × 1 day and 8 hours or 8 × 5 ounces, etc.

This activity should be liberally punctuated with discussion of examples as headlines and stories that fit them. Otherwise, there will be a tendency to follow computational rules rather than thinking out sensible and accurate conclusions.

7 × $7.13 = _____

"Mr. Bowen spent a week at a motel. His room cost $7 per night and there was 13¢ tax. His bill for 7 nights was $49.91."

"Each of a group of 7 girls received $7.13 when the profits of their week-long bake sale were divided evenly among them. Their total profits must have been $49.91."

"Alec Marson bought tickets for himself and his six sisters to take the inter-galactic shuttle to an asteroid. Each ticket was $7.13. He paid for them with a $50 bill and got 9¢ change."

"At a sale, a rack of blouses were specially priced at $2.64 each and a rack of skirts were priced at $4.49. Mrs. Larson bought 7 outfits and paid for them with the fewest bills and coins. She paid with 4 ten dollar bills, 1 five dollar bill, 4 one dollar bills, 1 half-dollar, 1 quarter, 1 dime, 1 nickel and 1 penny."

4 × 8 ft., 11 in. = _____

"A square room measured 8 feet and 11 inches on a side. Harry wants to put a strip of moulding along each side. A doorway will reduce the amount needed by 3 feet. He will need 32 feet, 44 inches less 3 feet, or 10 yards, 2 feet and 8 inches."

"Phyllis needs four pieces of material 8 feet, 11 inches long to make drapes for a window. She will need 11 yards, 2 feet and 8 inches, so she will buy 12 yards at 80¢ per yard. It will cost her $9.60, and she will get 40¢ change from a $10 bill."

You may want to ask the class to select a group of the best stories and restate them in terms of questions. Assure them that these questions will be given as a quiz in the next few days. At that time, they will have to write the headlines and give sensible answers to the questions.

An important pattern

Not obvious at first, is a pattern that runs through *Work Text* pages 30 through 33. With very few exceptions, each group of three examples has a certain interesting aspect to it.

The first three lines of example A on page 30 are essentially the following statements:

$2 \times \$7.13 = \$14.26$
$7 \times \$7.13 = \$49.91$
$9 \times \$7.13 =$ _____

Now, there are two ways in which that final amount can be computed. It can be considered a straightforward example: $9 \times 7.13 = 64.17$. However, an alert student might prefer to add:

$$\begin{array}{r} \$14.26 \\ 49.91 \\ \hline \$64.17 \end{array}$$

The pattern at work here, we hope, will become more and more familiar:

$9 \times 7.13 = (2 + 7) \times 7.13 =$
$(2 \times 7.13) + (7 \times 7.13)$

It is Ben's "Do-It-to-Both" rule developed in the *Level D Work Text*. It is what mathematicians call the "distributive principle." It is the pattern that makes the procedures of complex multiplication and division possible. The idea or pattern is expressed in the language of algebra as:

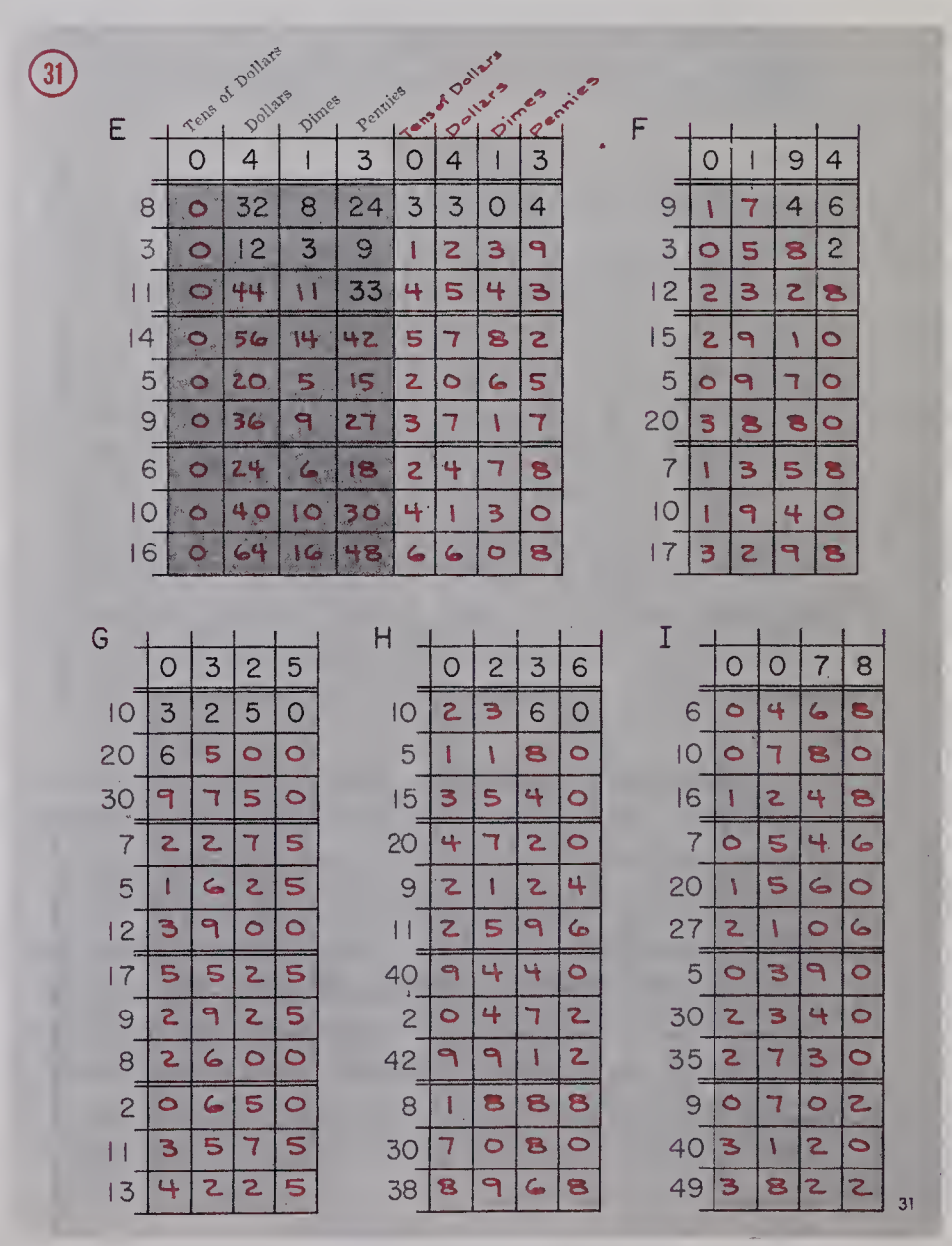
$$a\,(b + c) = (a \times b) + (a \times c)$$

We follow this pattern when we compute 7×13:

$$\begin{array}{r} 13 \\ \times\ 7 \\ \hline 91 \end{array} \quad \longrightarrow \quad \begin{array}{r} 10 + 3 \\ \times\quad 7 \\ \hline 70 + 21 \end{array} \quad \longleftarrow \quad 91$$

or

or $7 \times 13 = 7 \times (10 + 3) = 70 + 21 = 91$



32

Mrs. Esser said, "I call this game
SMALLEST TOTAL OF UNITS
Can you discover the rules and put them to work?"

	Weeks	Days	Hours				Gallons	Quarts	Pints				
	0	1	8	0	2	7	0	0	1	0	2	9	6
2	0	2	16	0	4	14	0	0	2	0	4	18	12
	0	2	16	1	2	2	0	1	0	0	5	9	2
3	0	3	24	0	6	21	0	0	3	0	6	27	18
	0	4	0	2	1	9	0	1	1	0	8	8	8
5	0	5	40	0	10	35	0	0	5	0	10	45	30
	0	6	16	4	0	11	0	2	1	1	4	8	0
4	0	4	32	0	8	28	0	0	4	0	8	36	24
	0	5	8	3	1	4	0	2	0	1	1	8	4
6	0	6	48	0	12	42	0	0	6	0	12	54	36
	1	1	0	5	0	6	0	3	0	1	7	7	6
10	0	10	80	0	20	70	0	0	10	0	20	90	60
	1	6	8	8	1	10	1	1	0	2	9	6	0
7	0	7	56	0	14	49	0	0	7	0	14	63	42
	1	2	8	6	0	1	0	3	1	2	0	7	2
8	0	8	64	0	16	56	0	0	8	0	16	72	48
	1	3	16	6	2	8	1	0	0	2	3	6	8
15	0	15	120	0	30	105	0	0	15	0	30	135	90
	2	6	0	12	2	9	1	3	1	4	4	4	0
9	0	9	72	0	18	63	0	0	9	0	18	81	54
	1	5	0	7	2	3	1	0	1	2	6	6	4
24	0	24	192	0	48	168	0	0	24	0	48	216	144
	4	4	0	20	2	0	3	0	0	7	1	0	4

32

33

	Hours	Minutes	Seconds			Tens of Dollars	Dollars	Dimes	Pennies				
	0	10	12	0	5	0	4	2	8	0	3	7	9
3	0	30	36	0	15	0	12	6	24	0	9	21	27
	0	30	36	0	15	1	2	8	4	1	1	3	7
5	0	50	60	0	25	0	20	10	40	0	15	35	45
	0	51	0	1	9	2	1	4	0	1	8	9	5
8	0	80	96	0	40	0	32	16	64	0	24	56	72
	1	21	36	2	8	3	4	2	4	3	0	3	2
2	0	20	24	0	10	0	8	4	16	0	6	14	18
	0	20	24	0	10	0	8	5	6	0	7	5	8
7	0	70	84	0	35	0	28	14	56	0	21	49	63
	1	11	24	2	3	2	9	9	6	2	6	5	3
9	0	90	108	0	45	0	36	18	72	0	27	63	81
	1	31	48	2	13	3	8	5	2	3	4	1	1
4	0	40	48	0	20	0	16	8	32	0	12	28	36
	0	40	48	1	4	1	7	1	2	1	5	1	6
6	0	60	72	0	30	0	24	12	48	0	18	42	54
	1	1	12	1	14	2	5	6	8	2	2	7	4
10	0	100	120	0	50	0	40	20	80	0	30	70	90
	1	42	0	3	2	4	2	8	0	3	7	9	0
16	0	160	192	0	80	0	64	32	128	0	48	112	144
	2	43	12	5	0	6	8	4	8	6	0	6	4
14	0	140	168	0	70	0	56	28	112	0	42	98	126
	2	22	48	4	6	5	9	9	2	5	3	0	6
19	0	190	228	0	95	0	76	38	152	0	57	133	171
	3	13	48	5	15	8	1	3	2	7	2	0	1

In division, we use a reverse version of the same pattern. In the language of algebra, it might be expressed as:

$$(m + n) \div r = (m \div r) + (n \div r)$$

We follow this pattern when we compute $97 \div 7$:

$$7\overline{)91} \longrightarrow 7\overline{)70+21}$$

$$\begin{array}{r} 3 \\ 10 \\ 7\overline{)91} \\ 70 \\ \hline 21 \\ 21 \\ \hline \end{array}$$

$10 + 3 = 13$

or $91 \div 7 = (70 \div 7) + (21 \div 7) =$
$10 + 3 = 13$

In traditional programs for teaching arithmetic, this basic idea is often glossed over or, as often, never mentioned at all. Without a clear grasp of this pattern, children simply cannot understand why the procedures of multiplication and division work. They have learned that a certain prescribed series of steps combining multiplication and division leads to a correct answer if performed in the right order and with accuracy. But there is no real understanding.

The lack of appreciation of the fundamental importance of this distributive principle or this "Do-It-to-Both" pattern has been one of the greatest contributing factors to distaste for arithmetic. Very few pupils can long enjoy performing tasks they do not understand.

Later, an extension of this pattern is fundamental to understanding algebra. Many teachers have found that before they can move ahead in algebra, they must go back and reteach that part of arithmetic in which adding and multiplication become intertwined. However, if arithmetic was learned without understanding this idea, reteaching is most unrewarding—if it is possible at all. Once lack of understanding has led to an increasing lack of interest, the task of reviving that interest is most formidable.

Each of these four *Work Text* pages, 30 through 33, keeps coming back to the combination of multiplication and addition: each third row can (with one or two exceptions) be completed either as a new problem in multiplication or simply as addition or subtraction of the two lines above.

Example A on *Work Text* page 30 reveals three instances of this pattern:

$2 \times \$7.13 =$	$5 \times \$7.13 =$	$10 \times \$7.13 =$
$7 \times \$7.13 =$	$3 \times \$7.13 =$	$4 \times \$7.13 =$
$9 \times \$7.13 =$	$8 \times \$7.13 =$	$6 \times \$7.13 =$

The last group of three offers a slight twist. The third line can be found by multiplication or by subtracting the result of the second line from the first line.

$$\begin{aligned} 6 \times \$7.13 &= \\ (10 - 4) \times \$7.13 &= 10 \times \$7.13 - 4 \times \$7.13 \\ &= \$71.30 - \$28.52 \\ &= \$42.78 \end{aligned}$$

In algebra, this pattern might be expressed as:

$r\,(s - t) = (r \times s) - (r \times t)$.

Example B on page 30 provides 3 more instances of this idea:

$10 \times 684 =$	$7 \times 684 =$	$5 \times 684 =$
$6 \times 684 =$	$2 \times 684 =$	$3 \times 684 =$
$4 \times 684 =$	$9 \times 684 =$	$8 \times 684 =$

If some of the children do not see this plan built into these four pages without any help, then their attention ought to be drawn to it.

Point out the fact that line 3 of example A on page 30 could be completed by adding the results in the lines above it—treating it as if it were a coincidence. Later on, point out another similar relationship. Later on still, see if anyone can find another set of examples with this relationship in some form.

Soon there will be many volunteers pointing out instances. Eventually it will become obvious that these are not coincidences . . . in fact, the double horizontal lines on pages 30 and 31 are hard to explain unless each sets off an example of this pattern.

Emphasis on this underlying repetition is important preparation for developments that follow immediately in which we are going to study this idea in depth.

Notes to accompany *Work Text* pages 34-37

"Take a brief look at page 33. In the next two minutes, see how many examples you can complete in the area that is shaded with green. Go!"

Keep extending the two minutes unless some of the children complete all the examples. As soon as some finish: "Stop! Take a look at the examples in the whole area. In the next four minutes, see how many examples you can complete in the unshaded area. Go!"

Again, keep extending the time until some finish. "Stop! Is there a familiar pattern on this page?"

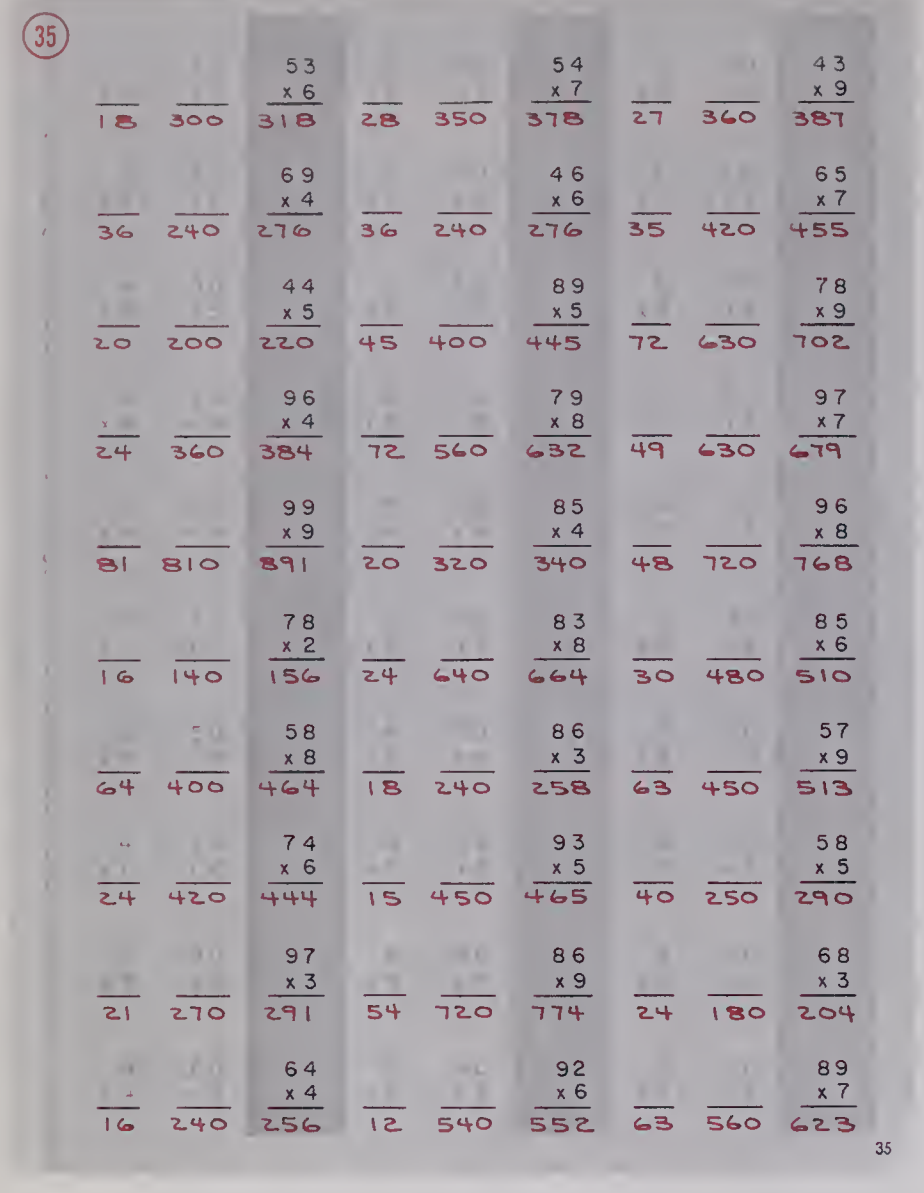
The purpose of this question is not to prompt attempts at accurately describing that pattern. Look back at the four previous pages. The pattern was uncovered there in each group of three examples.

"Is there a somewhat similar pattern in each group of three examples on *Work Text* page 34?" Put the first group on the board, along with another from the central portion of the page:

$3 \times 2 = 6$	3	10	13
$3 \times 10 = 30$	$\times\ 4$	$\times\ 4$	$\times\ 4$
$3 \times 12 = 36$	12	40	52

Before discussion gets more than barely underway, direct attention back to *Work Text* page 30. "Perhaps recognizing that the examples are arranged in groups of 3 will suggest some shortcuts you can use in completing the page."

Everyone should work on his own.



Variations of this introduction of page 30 should be used with the three pages that follow—31, 32 and 33. Throughout the work on these four pages, discussions should be avoided. It is most important that each child have a firm grasp of the idea we are spotlighting before any words get attached to it. There is much to discover in this activity.

Even those who are already proficient in computing answers to problems of multiplication, will find themselves persuaded to take shortcuts. Some of the faster computers, perhaps, never quite understood why they did what they did.

Whenever you are tempted to talk in detail about this pattern mathematicians call the "distributive principle" and Ben calls his "Do-It-to Both rule," remember that the next 20 *Work Text* pages are going to keep coming back to this idea again and again in several different ways. Discussion can wait —until every child has the idea firmly fixed in his own mind. At that time, names are immaterial and will not warp the idea because it is no longer in the early formative stages.

Headline stories

Activity around these four pages needs considerable use of the headline-story technique to help children.

$$\begin{array}{r} 3 \times 2 = 6 \\ 3 \times 10 = 30 \\ \hline 3 \times 12 = 36 \end{array}$$

"I bought 3 pencils at 10¢ each and erasers for each at 2¢. I spent 36¢ . . . because the erasers cost a total of 6¢ and the pencils a total of 30¢—and 6¢ plus 30¢ is 36¢; or, because each pencil and eraser combination cost 12¢ and 3 times 12¢ is 36¢."

"There were 2 pilots and 10 helpers on each of 3 space ships. There were 36 crewmen in all—6 pilots and 30 helpers. On each space ship there was a crew of 12; on 3 space ships, 3×12 or 36 crewmen."

"In each of 3 rows, Mrs. Jones planted 2 geraniums and 10 daisies. That's a total of 36 plants—6 geraniums and 30 daisies. Or you can see it as 3 rows with 12 plants in each row, and $3 \times 12 = 36$."

$$\begin{array}{r} 5 \times 7 = 35 \\ 5 \times 20 = 100 \\ \hline 5 \times 27 = 135 \end{array}$$

"Alice had a pile of 7 nickels and another pile of 20 nickels. One was worth 35¢ and the other was worth $1.00, or $1.35 altogether."

"There are 7 more weeks of school before Christmas, or 35 days; and there are 20 more weeks of school after that before vacation, or 100 days of school. There are 135 days of school before I leave for my trip to the moon. That's the same as 27 weeks of school days, and $5 \times 27 = 135$."

6	90	96
x 4	x 4	x 4
24	360	384

"Jennie bought three theatre tickets at 90¢ and had to pay 6¢ tax on each. Of the $3.84 she spent, 24¢ went for tax and $3.60 for tickets. Four tickets (tax included) cost 96¢ each, or a total of $3.84. So, Jennie got $1.16 change from her $5 bill."

"A box containing playground equipment weighed 96 pounds. Four such boxes were delivered to the school. Shipping costs were 25¢ per pound. So, the cost was 384×25¢, or $9.60. Each box weighed 6 pounds, and the equipment in each weighed 90

a.	b.	c.
x 1 = 6	6 x 3 = 18	4 x 5 = 20
x 10 = 30	6 x 20 = 120	4 x 20 = 80
x 300 = 900	6 x 100 = 600	4 x 300 = 1,200
x 312 = 936	6 x 123 = 738	4 x 325 = 1,300

5 x 5 = 25	20 x 5 = 100	300 x 5 = 1,500	325 x 5 = 1,625	l.	7 x 9 = 63	00 x 9 = 0	400 x 9 = 3,600	407 x 9 = 3,663
2 x 2 = 14	10 x 2 = 20	500 x 2 = 1,000	517 x 2 = 1,034	m.	0 x 7 = 0	90 x 7 = 630	800 x 7 = 5,600	890 x 7 = 6,230
8 x 5 = 40	40 x 5 = 200	700 x 5 = 3,500	748 x 5 = 3,740	n.	9 x 6 = 54	40 x 6 = 240	700 x 6 = 4,200	749 x 6 = 4,494
6 x 4 = 24	40 x 4 = 160	800 x 4 = 3,200	846 x 4 = 3,384	o.	1 x 8 = 8	90 x 8 = 720	200 x 8 = 1,600	291 x 8 = 2,328
8 x 6 = 48	50 x 6 = 300	100 x 6 = 600	158 x 6 = 948	p.	4 x 7 = 28	00 x 7 = 0	600 x 7 = 4,200	604 x 7 = 4,228
9 x 3 = 27	40 x 3 = 120	800 x 3 = 2,400	849 x 3 = 2,547	q.	7 x 8 = 56	40 x 8 = 320	600 x 8 = 4,800	647 x 8 = 5,176
6 x 5 = 30	50 x 5 = 250	200 x 5 = 1,000	256 x 5 = 1,280	r.	0 x 9 = 0	10 x 9 = 90	200 x 9 = 1,800	210 x 9 = 1,890
8 x 8 = 64	50 x 8 = 400	300 x 8 = 2,400	358 x 8 = 2,864	s.	9 x 9 = 81	50 x 9 = 450	300 x 9 = 2,700	359 x 9 = 3,231

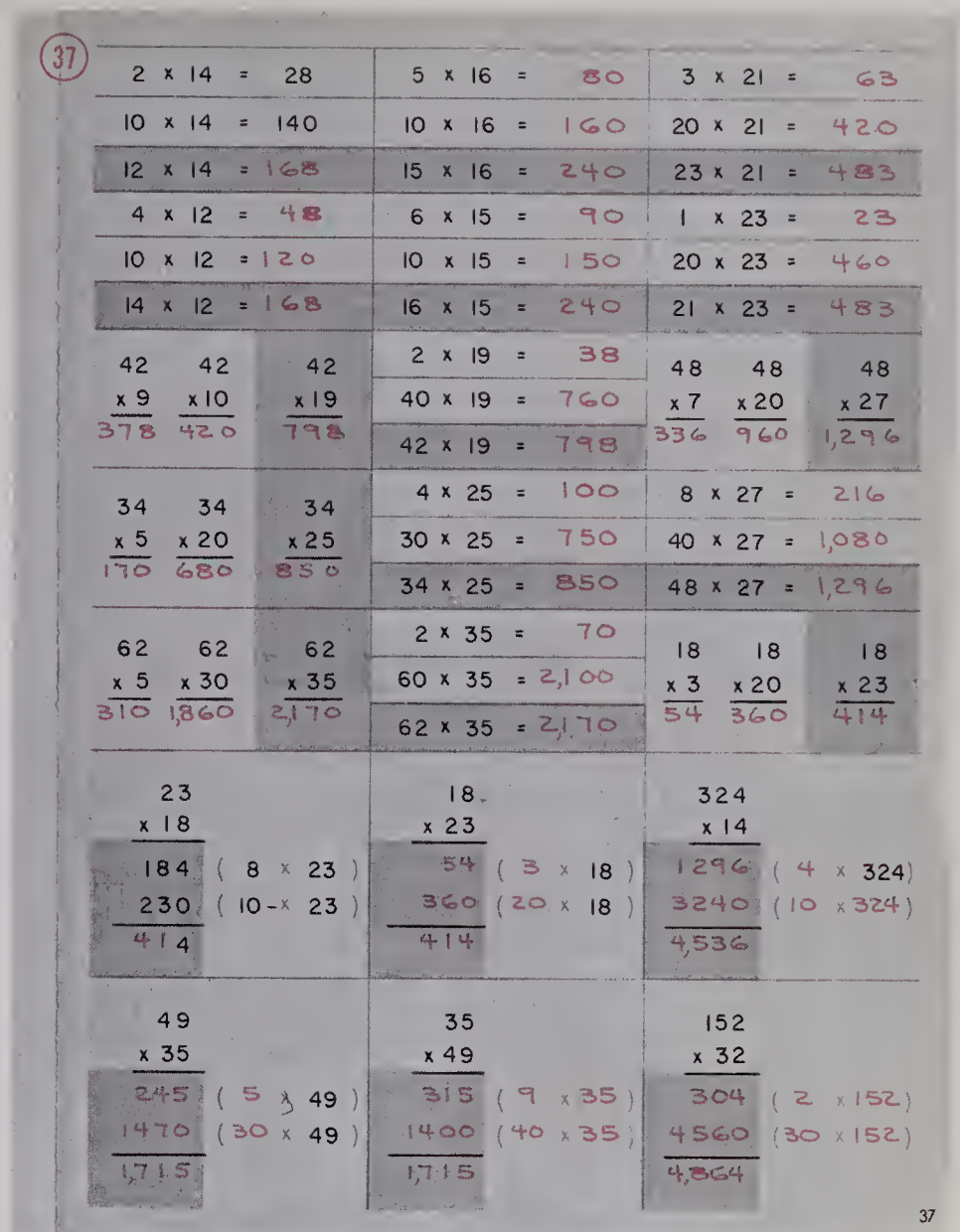
pounds. So, there was 24 pounds of boxes and 360 pounds of equipment in all."

"Mr. Boyles needed 4 bags of lime at 6¢ per bag delivered. Lime costs only 90¢ a bag if you pick it up. He had planned to spend $3.84 for lime, but decided to save 24¢ by picking up the 4 bags. So, when he handed the clerk $4.00, he received 40¢ change."

4	x	25	= 100
30	x	25	= 750
34	x	25	= 850

"Bill added four quarters to his savings collection of 30 quarters. It was worth $7.50 before, but now it's worth a dollar more. His 34 quarters are worth $8.50."

"Each of the 25 tanks on the rocket was filled with 30 gallons of fuel, or 750 gallons in all. The project director decided that 100 more gallons were needed, so he put 4 more gallons in each of the 25 tanks. Now there were 34 gallons in each tank, or 850 gallons in all . . . enough to achieve the escape velocity of more than 25,000 miles per hour."



Divide the class into four or six teams. Each team can select any five groups of three related examples from these four *Work Text* pages. They are to write an interesting story to go with each. When they are finished, each team reads its stories. Then the class can decide which team came up with the most interesting—which will be "a quiz for the whole class tomorrow." For the quiz, the problem is to write an appropriate headline for each.

The idea we are developing is so fundamental to understanding arithmetic that children cannot spend too much time in searching for examples of it in the world around them.

Another pattern

Work Text page 37 emphasizes another important pattern. It is a pattern we often use in checking computations. We also use it as a shortcut. In algebra, it is expressed simply as: $a \times b = b \times a$.

Most of the examples on page 37 are in this form.

$12 \times 14 = 168$	$15 \times 16 = 240$
$14 \times 12 = 168$	$16 \times 15 = 240$

Mathematicians call this the "commutative principle"—others refer to it as part of the "in any order

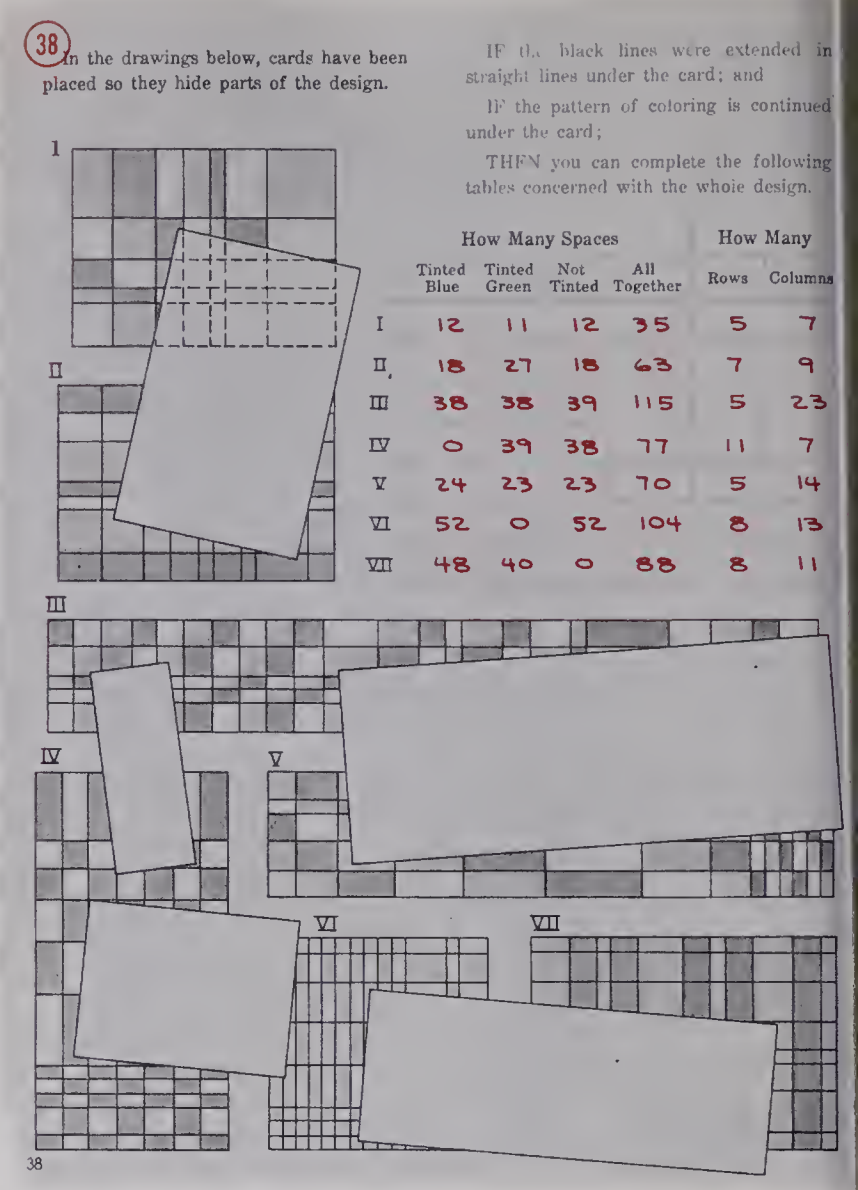
rule" for multiplication. It is a principle that reduces the 100 basic multiplication facts to only slightly more than 50:

$$3 \times 7 = 7 \times 3 = 21$$
$$9 \times 6 = 6 \times 9 = 56$$
$$8 \times 5 = 5 \times 8 = 40$$

When confronted by computing the product of 44 and 97, we prefer the first of these two:

97	44
× 44	× 97
388	308
388	396
4268	4268

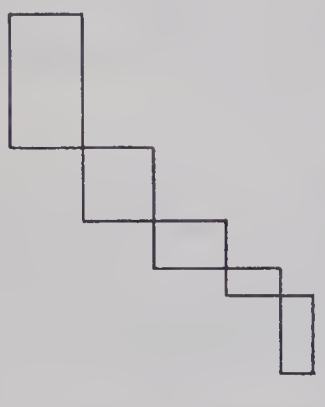
This idea is so obvious that most children understand it adequately. We hope that children will become as familiar with the "distributive principle" that is the main focus of attention in these pages.


38
In the drawings below, cards have been placed so they hide parts of the design.
IF the black lines were extended in straight lines under the card; and
IF the pattern of coloring is continued under the card;
THEN you can complete the following tables concerned with the whole design.
How Many Spaces
How Many
Tinted Blue
Tinted Green
Not Tinted
All Together
Rows
Columns
I 12 11 12 35 5 7
II 18 27 18 63 7 9
III 38 38 39 115 5 23
IV 0 39 38 77 11 7
V 24 23 23 70 5 14
VI 52 0 52 104 8 13
VII 48 40 0 88 8 11
I
II
III
IV
V
VI
VII
38

Notes to accompany *Work Text* pages 38-39

Pages 38 and 39 are directed to the study of a certain kind of patterns. The pattern in example I is partially blocked out—at least the color is missing though the extensions of lines is indicated. "Is there a clue in the portion shown that tells the way in which the rest of the design is probably colored?"

In all the portion shown, each color follows a down-right diagonal—blue, green and white:



So, by extending this, the colors of the hidden blocks can be determined (always assuming the visible pattern was extended).

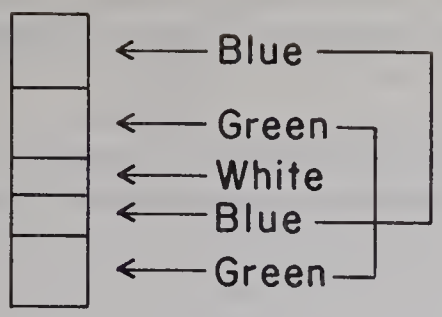
Another means to determine colors of the covered section is to see that each row follows a consistent left to right rotation—blue, green, white, blue, green, white, blue, etc. Similarly, each vertical row, from top to bottom, fits into this sequence—blue, white, green, blue, white, green, blue, etc.

In examples II through XVII, it is important to remember the explanation at the top of the page—the black line separations are extended in straight lines under the cards.

All the information needed in II is: (a) there are 9 blocks in each row; (b) 3 rows are green; (c) 2 rows are blue; and (d) 2 are white. So, there are 3×9 green blocks, 2×9 blue blocks, and 2×9 white ones. In all, there are $(3 + 2 + 2) \times 9$ or 7×9 blocks—63 in all.

In example 3, each row, left to right, and each column, top to bottom, follows a part of this sequence: green, white, blue, green, etc. Or, each down-left diagonal repeats the same color, and the diagonals, from left to right, follow the sequence noted above.

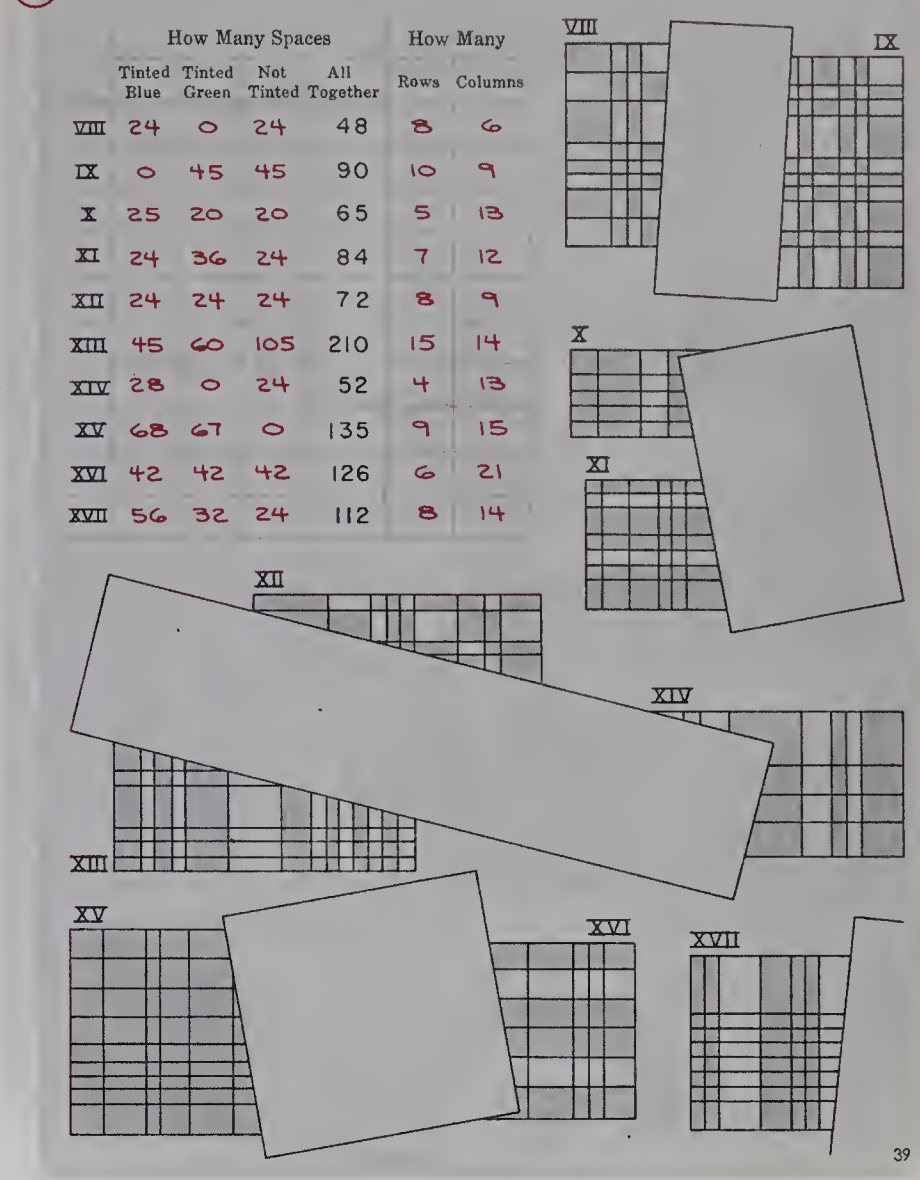
One interesting way to proceed is to notice that each row has 23 blocks. If each had 24, then it would contain 8 of each color. There are 5 rows, so there would be 40 of each. Now, if you can imagine an additional column on the left side, it would be:



If this column were added, there would be 40 of each. So there must be 40 − 2 blue, 40 − 2 green, 40 − 1 white—38 blue, 38 green, 39 white. Of course, there are other ways of unraveling each puzzle. The other examples on page 38 can be solved by similar means.

On *Work Text* page 39, reference must be made to the charts that give the total number of blocks . . . 48 in example VIII. Inspection of the design shows there are 8 rows. So, each row must have 6 blocks in it because 48 ÷ 8 = 6 or 6 × 8 = 48.

Further inspection of the columns shows that there are an equal number of blue and untinted blocks in each. So, in all, there must be 24 blue, 24 untinted—and 0 green blocks.



In example IX there are 10 blocks in each column, so there must be 9 in each row—the chart says there are 90 in all. And, once again, inspection shows an equal number of white and green in each column.

Example X shows 5 blocks in a column. The chart says there are 65 in all; so there must be 13 columns. Since the color rotation in the column is blue, green, white, blue—it must continue to green, white, blue, green, etc. Each 3 columns repeats the scheme. In 12 columns there will be 4 repetitions of blue, green, white. Therefore, there must be 4 columns of each color with 5 blocks in, or 20 of each in the first 12 columns. The thirteenth column would be blue. So there must be 20 green, 20 white and 25 blue blocks.

The last example

The last example requires a bit of study. There are 8 blocks in each column and 112 (see chart) in all; so there are 112 ÷ 8 or 14 columns.

A careful study shows that the pattern repeats itself every 4 columns:

GBWB GBW*B* *GBWB* *GB*

there are 3 + 1 green, or 4 green
6 + 1 blue, or 7 blue
3 white

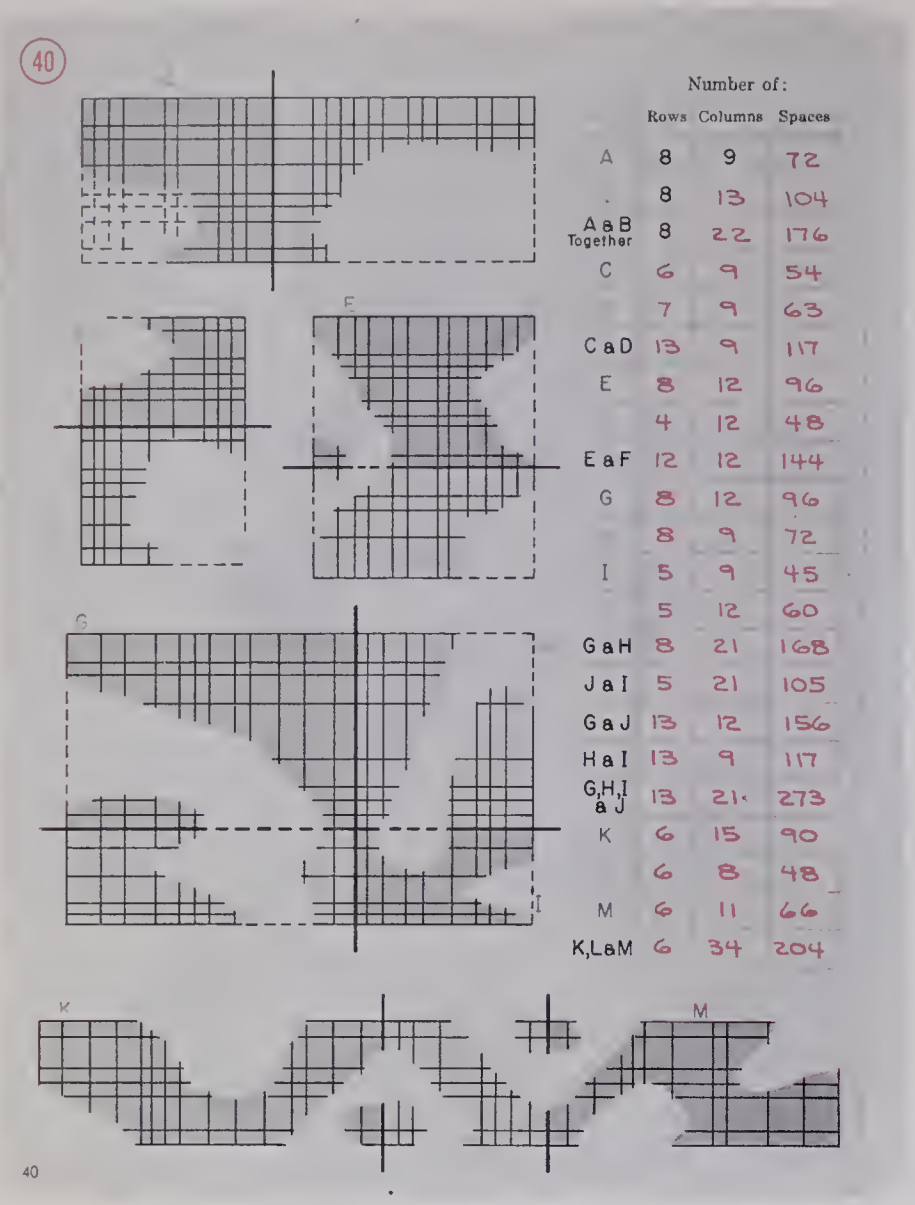
Since there are 8 blocks in each column, there must be

4 × 8 or 32 green
7 × 8 or 56 blue
3 × 8 or 24 white
112 total

In Level D, Ben would say that this is an example of his "Do-it-to-all three" rule. He would write it this way:

$$8 \times 14 = 8 \times (4 + 7 + 3) = 8 \times 4 + 8 \times 7 + 8 \times 3$$
$$= 32 + 56 + 24$$
$$= 112$$

When these patterns are analyzed in terms of rows and columns, the results usually employ the same "distributive principle" or a variation of it. We will pursue this idea again and again, going deeper and deeper as this section of pages develops.



Notes to accompany *Work Text* pages 40-41

On page 40, each rectangular design is a solid color, so they need less careful study. However, in a few examples, the black lines may need to be extended to find the number of rows and columns.

Each of the five combination designs illustrate a pattern that can be expressed neatly in a mathematical expression:

A&B: $8 \times 22 = 8\,(9 + 12) =$
$8 \times 9 + 8 \times 13 = 176$

C&D: $13 \times 9 = (6 + 7)\,9 =$
$6 \times 9 + 7 \times 9 = 117$

E&F: $12 \times 12 = (8 + 4)\,12 =$
$8 \times 12 + 4 \times 12 = 144$

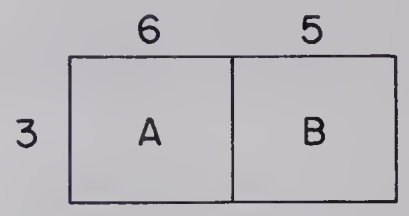
H&L&M: $6 \times 34 = 6\,(15 + 8 + 11) =$
$6 \times 15 + 6 \times 8 + 6 \times 11 = 204$

The design combining G, H, I & J is the same idea, but more complex. Ben, Level D, called this his "Double-Do-It-to-Both" rule.

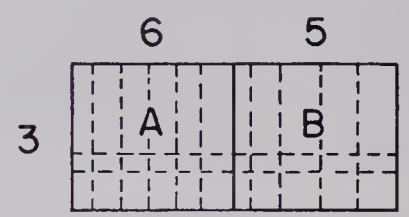
$$\begin{aligned} 13 \times 21 &= (8 + 5) \times (12 + 9) \\ &= 8 \times 12 + 8 \times 9 + 5 \times 12 + 5 \times 9 \\ &= 96 + 72 + 60 + 45 \\ &= 273 \end{aligned}$$

This example need not be developed at great length —because we return to the idea on *Work Text* page 42.

The idea in the first three examples on page 40 is explored further on page 41. Now the lines are not drawn to indicate the blocks. Another device is used to convey this information. For example, a chart provides enough clues, associated through different shaped boxes, to describe the pattern involved. On the board, draw a sketch to illustrate the first example. The information given is this:

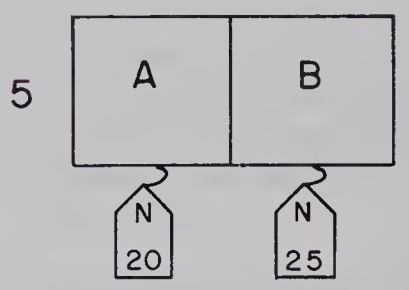


With lines drawn in, this becomes



$$\begin{aligned} A &\rightarrow 3 \times 6 = 18 \\ B &\rightarrow 3 \times 5 = 15 \\ \hline A \,\&\, B &\rightarrow 3 \times 11 = 33 \end{aligned}$$

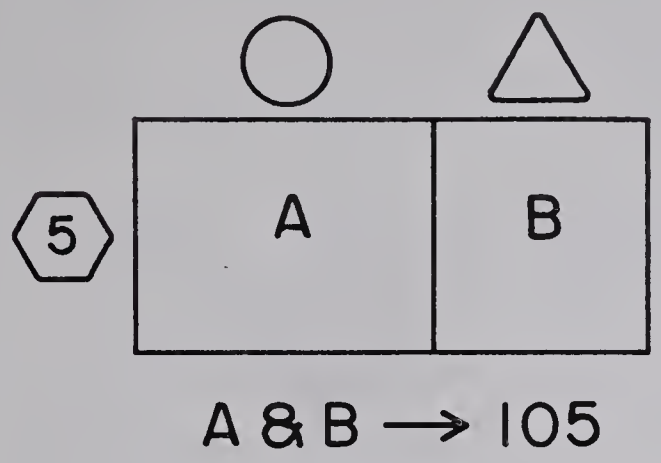
For the second example, different information is given:



If there are 20 blocks in 5 rows in A, there must be 4 in each row in A. In each of the 5 rows in B, there must be 5 blocks, since there are 25 in all.

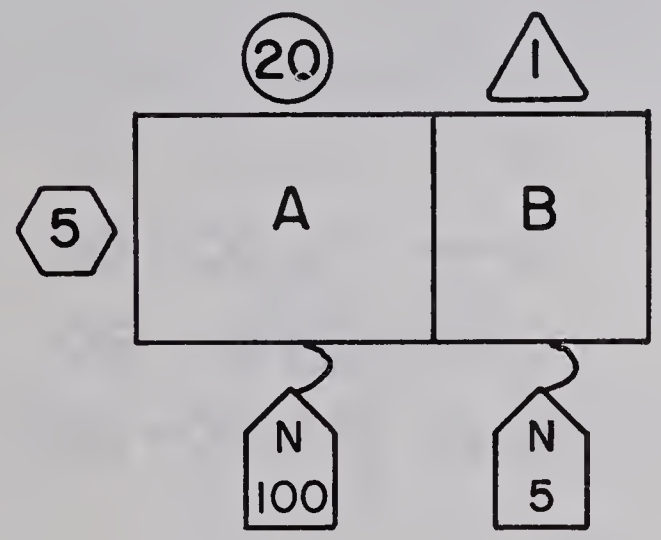
$$\begin{aligned} A &\rightarrow 5 \times 4 = 20 \\ B &\rightarrow 5 \times 5 = 25 \\ A \,\&\, B &\rightarrow 5 \times 9 = 45 \end{aligned}$$

Throughout these problems, sketches will help toward solutions. A bit of the puzzle is introduced in examples VI through XXV. Two rules reserve the circle for multiples of 10, and the triangle for numbers less than 10. Example VII looks like this when all given information is indicated:



A & B → 105

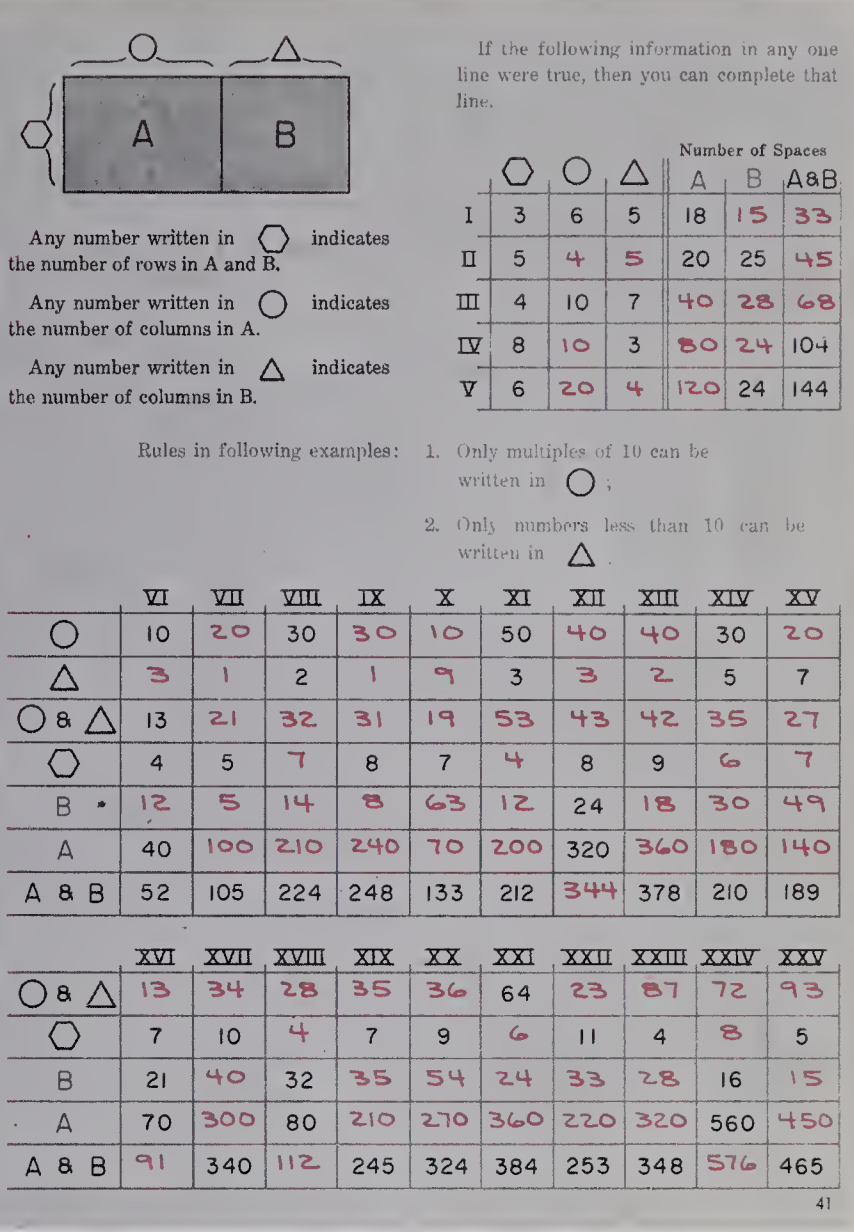
In each row all the way across there must be 21 blocks. So ○ + △ → 21; and only multiples of 10 can be written in ○ and numbers less than 10 in △. The answer must be that 20 is written in the circle and 1 in the triangle. So:



A → 100
B → 5
A&B → 105

With the help of sketches, enough information is given in the examples to reason through to a reasonable conclusion. Each will be a sketch of an example of multiplication in which the regrouping follows the pattern of the traditional algorism:

21 × 5 → 105 ←--- 20 + 1 × 5 → 100 + 5



41

Any number written in ⬡ indicates the number of rows in A and B.

Any number written in ○ indicates the number of columns in A.

Any number written in △ indicates the number of columns in B.

If the following information in any one line were true, then you can complete that line.

	⬡	○	△	Number of Spaces A	B	A&B
I	3	6	5	18	15	33
II	5	4	5	20	25	45
III	4	10	7	40	28	68
IV	8	10	3	80	24	104
V	6	20	4	120	24	144

Rules in following examples:
1. Only multiples of 10 can be written in ○;
2. Only numbers less than 10 can be written in △.

	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV
○	10	20	30	30	10	50	40	40	30	20
△	3	1	2	1	9	3	3	2	5	7
○ & △	13	21	32	31	19	53	43	42	35	27
⬡	4	5	7	8	7	4	8	9	6	7
B	12	5	14	8	63	12	24	18	30	49
A	40	100	210	240	70	200	320	360	180	140
A & B	52	105	224	248	133	212	344	378	210	189

	XVI	XVII	XVIII	XIX	XX	XXI	XXII	XXIII	XXIV	XXV
○ & △	13	34	28	35	36	64	23	87	72	93
⬡	7	10	4	7	9	6	11	4	8	5
B	21	40	32	35	54	24	33	28	16	15
A	70	300	80	210	270	360	220	320	560	450
A & B	91	340	112	245	324	384	253	348	576	465

41

or the division algorism:

$$5\overline{)105} \rightarrow 5\overline{)100+5} \rightarrow 5\overline{)105}$$

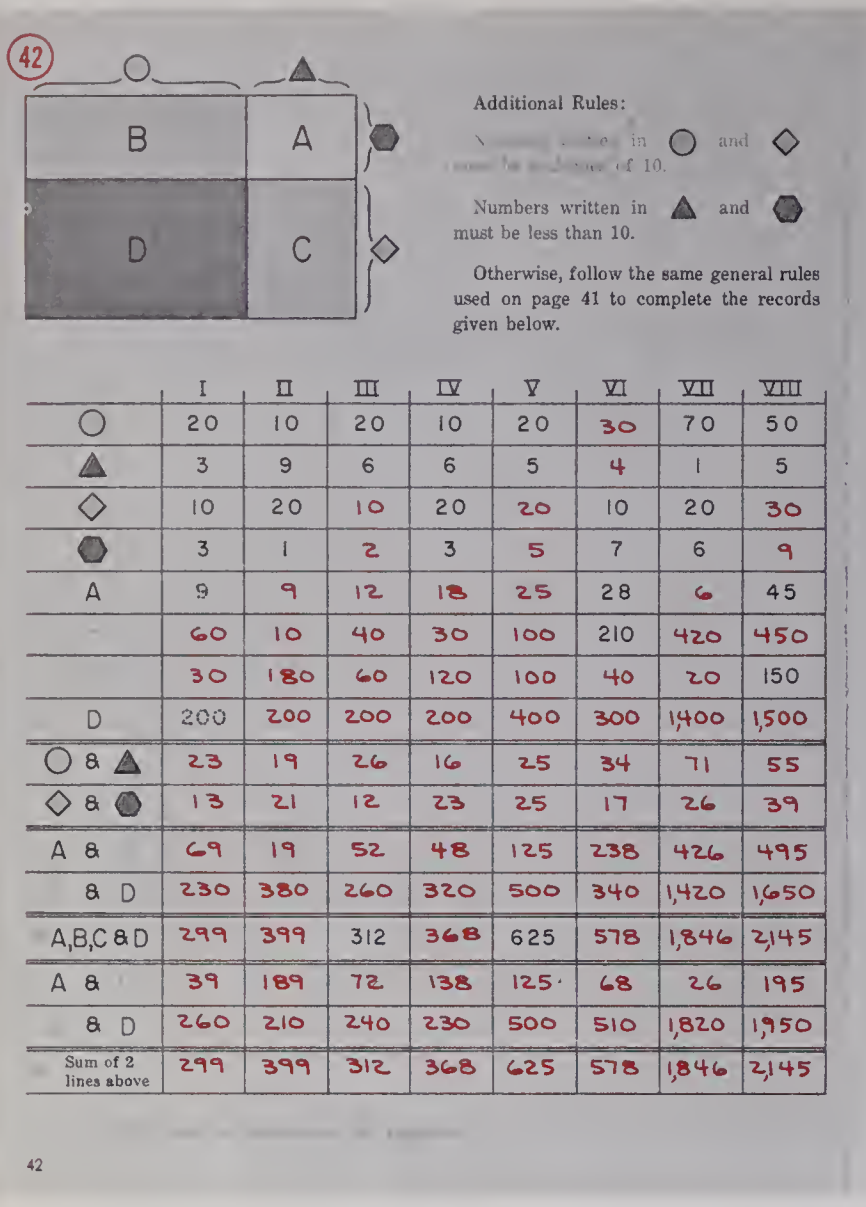
$$\begin{array}{r} 21 \\ 5\overline{)105} \end{array} \rightarrow \begin{array}{r} 20+1=21 \\ 5\overline{)100+5} \end{array} \rightarrow \begin{array}{r} 1 \\ 20 \\ 5\overline{)105} \\ 100 \\ \hline 5 \\ 5 \\ \hline \end{array}$$

However, postpone any such discussion. Let each child find out as much as he can for himself.

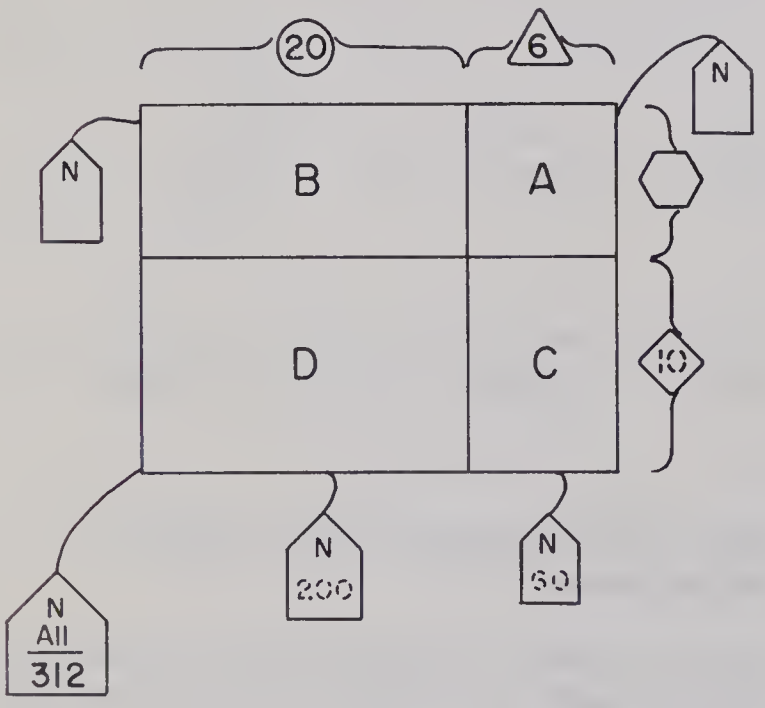
Notes to accompany *Work Text* pages 42-44

Pages 42, 43 and 44 develop further the idea introduced on page 40—which Ben, in Level D, called his "Double-Do-It-to-Both."

An extension of the rules on page 41 now reserves both the circle and the diamond (both tinted blue) for multiples of 10. The triangle and hexagon are reserved for numbers less than 10 (both tinted



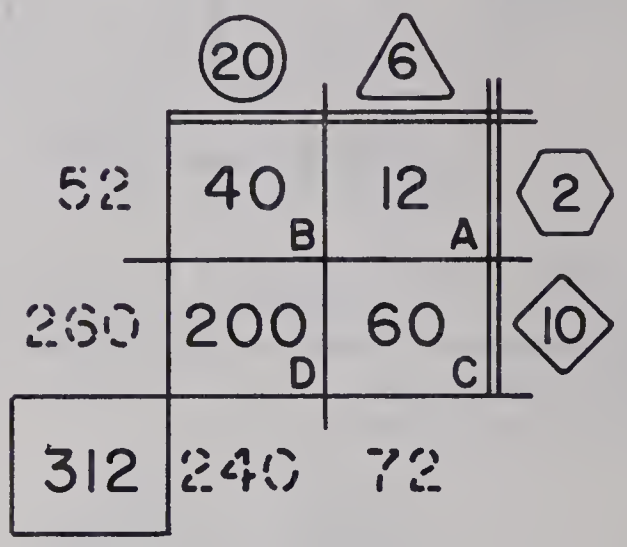
green). Each column in the chart gives clues about a different situation. Consider example III with a sketch indicating the information given:



If we put 20 in the diamond, then D alone would contain 20 × 20 blocks—or 400. But there are only 312 in all. It would seem more reasonable to put 10 in the diamond. This would mean that D would contain 10 × 20 or 200 and C would contain 10 × 6 or 60. Now we have accounted for 260 of the 312. There must be 52 unaccounted for, because 312 − 260 = 52.

If we try 1 in the hexagon, then there would be 20 in B and 6 in A or 26 in both together. But if we put 2 in the hexagon, then there would be 2 × 20 in B and 2 × 6 in A, or 40 + 12—and that is 52, the number we had to account for.

Here is another way we could keep track of these results:



A → 2 x 6 = 12

B → 2 x 20 = 40

C → 10 x 6 = 60

D → 10 x 20 = 200

At the left of the top row we can write 52 to indicate that there are 52 accounted for in the top row—A & B. At the left of the next row, we indicate that 260 are accounted for in D & C . . . and 52 + 260 = 312.

At the bottom of the left-hand column, we indicate that B & D account for 40 + 200 or 240. At the bottom of the right-hand column, we note that 72 are accounted for in A & C . . . and 240 + 72 = 312.

This is a nice kind of record—and we will use it often, beginning on *Work Text* page 44. In fact, Alec Marson has a machine we will study on the next page that is very similar.

Suppose, in the example above, we remove the restrictions about what can be written in the shapes. Consider example V:

Mr. Wilson, the principal, was visiting the class when they were working on page 42. He asked Mrs. Esser for an opportunity to explain an interesting development.

"You know," he began, "when I experimented with Alec Marson's machine, one of the change buttons produced a result similar to that found on page 42. Here are some of the records I kept of the experiments. Can you fill in the blanks?"

I write

A

36 × 54 = 1,944			
54 × 36	5	4	X
324	30	24	6
162	15	12	3
1,944	1,80	144	

B

52 × 68 = 3,536			
68 × 52	6	8	X
136	12	16	2
340	30	40	5
3,536	3,12	416	

C

46 × 92 = 4,232			
92 × 46	9	2	X
552	54	12	6
368	36	8	4
4,232	4,14	92	

D

27 × 138 = 3,726				
138 × 27	1	3	8	X
966	7	21	56	7
276	2	6	16	2
3,726	2,7	81	216	

36 × 54	52 × 68	46 × 92	216 × 23	375 × 41
144	416	92	648	375
180	312	414 0	432	1500
1,944	3,536	4,232	4,968	15,375

Because 0 always occurs where they are shown in blue, it is customary to omit them altogether.

44 Multiples of 10 in blue-tinted spaces: Numbers less than 10 in green spaces:

39 × 12	30	9	
78	60	18	2
390	300	90	10
468	360	108	

27 × 43	20	7	
81	60	21	3
1080	800	280	40
1,161	860	301	

21 × 34	20	1	
84	80	4	4
630	600	30	30
714	680	34	

19 × 51	10	9	
19	10	9	1
950	500	450	50
969	510	459	

36 × 58	30	6	
288	240	48	8
1800	1500	300	50
2,088	1740	348	

92 × 24	90	2	
368	360	8	4
1840	1800	40	20
2,208	2160	48	

57 × 86	50	7	
342	300	42	6
4560	4000	560	80
4,902	4300	602	

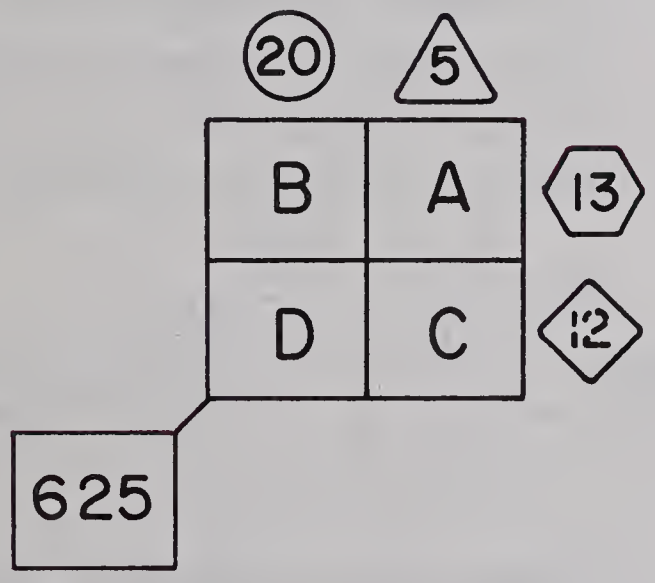
94 × 36	90	4	
564	540	24	6
2820	2700	120	30
3,384	3240	144	

112 × 34	110	2	
448	440	8	4
3360	3300	60	30
3,808	3740	68	

47 × 37	40	7	
329	280	49	7
1410	1200	210	30
1,739	1480	259	

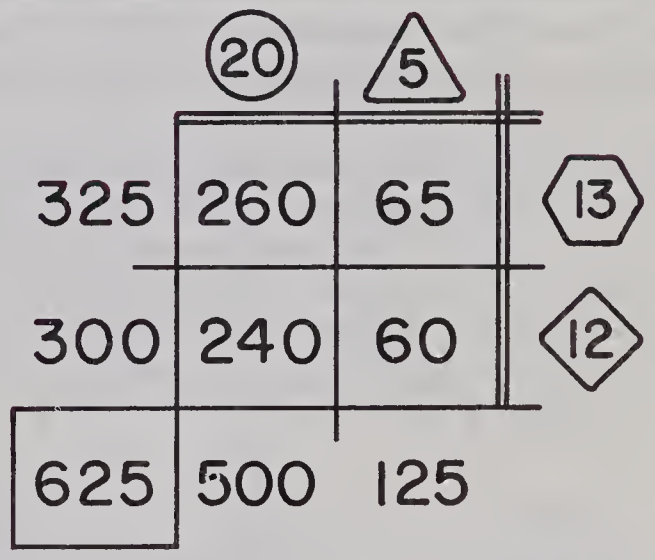
Complete	
37 × 23	
111	
740	
851	
67 × 45	
335	
2680	
3,015	
309 × 28	
2472	
6180	
8,652	
239 × 35	
1195	
7170	
8,365	
38 × 28	
304	
760	
1,064	
76 × 67	
532	
4560	
5,092	

44

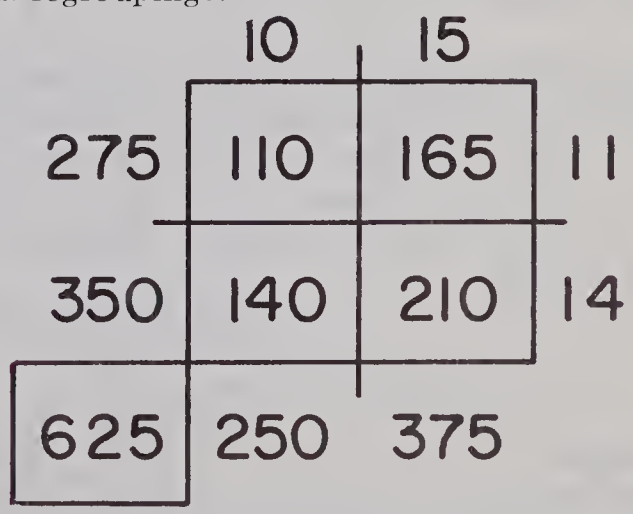


Try 13 in the hexagon. Now, we know there would be 13×5 blocks in A and 13×20 in B—and $65 + 260 = 325$. So, we have accounted for 325 of 625 and $625 - 325 = 300$. Since we have 300 to account for, we must write a number slightly less than 13 in the diamond, because 300 is less than 325.

Try 12 in the diamond. In C there will be 12×5 or 60; and in D there will be 12×20 or 240. Since $240 + 60 = 300$, we have now accounted for all 625. To summarize in a Cross Number Puzzle form:



Finally, consider this example of 25×25 with various regroupings:



We find that, as long as the sum of the numbers written in the circle and square is 25 and that the

45

If a product and one factor is known, you can discover quite a bit about the missing factors. (Leave green-tinted area until last.)

	Known Factor	Product	Multiples of Known Factor x 10	x 100	x 1000	Information about Missing Factor (If any) Number of digits	Digit on left	Next digit on right	Next digit to right
1.	8	768	80	800	8,000	2	9	6	none
2.	3	2,865	30	300	3,000	3	9	5	5
3.	5	1,400	50	500	5,000	3	2	8	0
4.	9	783	90	900	9,000	2	8	7	none
5.	3	783	30	300	3,000	3	2	6	1
6	11	396	110	1,100	11,000	2	3	6	none
7.	15	3,225	150	1,500	15,000	3	2	1	5
8.	45	2,025	450	4,500	45,000	2	4	5	none
9.	7	5,439	70	700	7,000	3	7	7	7
10.	27	783	270	2,700	27,000	2	2	9	none
11.	2	5,318	20	200	2,000	4	2	6	5
12.	19	11,533	190	1,900	19,000	3	6	0	7
13.	125	1,125	1,250	12,500	125,000	1	9	none	none
14.	36	16,416	360	3,600	36,000	3	4	5	6
15.	89	8,722	890	8,900	89,000	2	9	8	none
16.	48	27,648	480	4,800	48,000	3	5	7	6
17.	55	3,025	550	5,500	55,000	2	5	5	none
18.	91	7,826	910	9,100	91,000	2	8	6	none
19.	52	7,696	520	5,200	52,000	3	1	4	8
20.	72	47,952	720	7,200	72,000	3	6	6	6
21.	11	7,194	110	1,100	11,000	3	6	5	4
22.	56	27,944	560	5,600	56,000	3	4	9	9

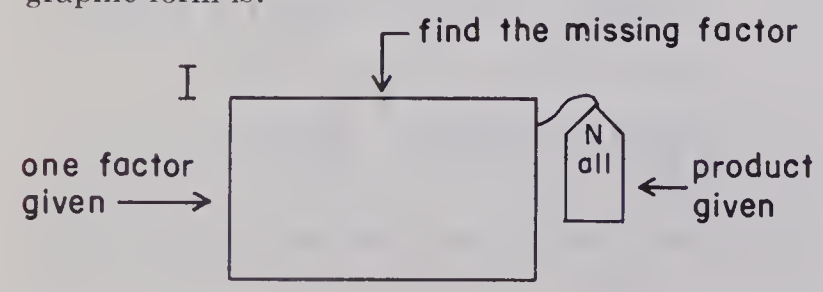
45

same is true of the numbers written in the hexagon and diamond, the grand total will always be 625.

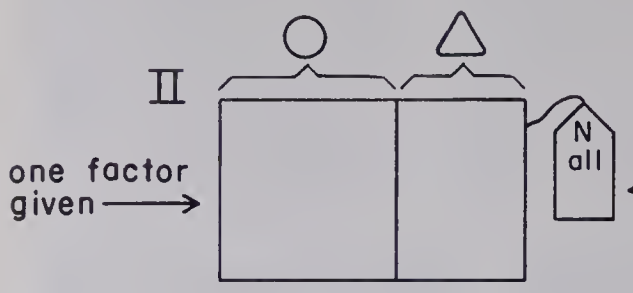
Let the children work independently on page 43—trying on their own to find out what Alec's machine does. Of course, it is following the same pattern we have been studying generally for several pages—and specifically on the preceding page. Page 44 is, by now, clearly understood and can be used as a check-up.

Notes to accompany *Work Text* pages 45-47

The next three pages—45, 46 and 47—consider the application in the division algorism of the basic idea we have been developing. In this case, a product and one of its factors is given—and we need to develop ways of finding the missing factor. The problem in graphic form is:



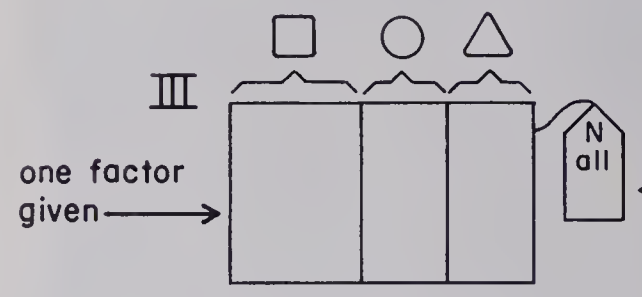
We find from experience that the problem often becomes more manageable if we make a two-step search out of it and impose on ourselves precisely the same rules that introduced the game element in previous pages. In graphic form we look at it this way:



○ Reserved for multiples of 10

△ Reserved for numbers less then 10

When the multiples of 10 are 100 or more, we stretch out our hunt into three steps and change the rules slightly:

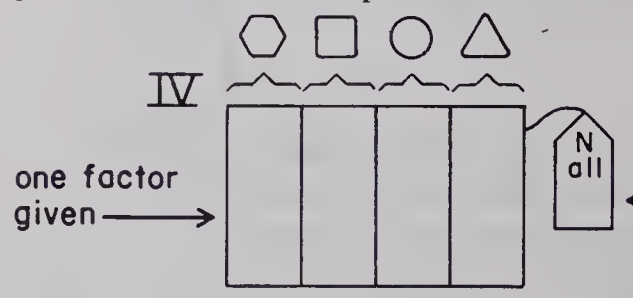


□ Reserved for multiples of 100

○ Reserved for multiples of 10

△ Reserved for numbers less then 10

When multiples of 100 are 1000 or more, we expand our sketch to keep the search manageable:



⬡ Reserved for multiples of 1,000

□ Reserved for multiples of 100

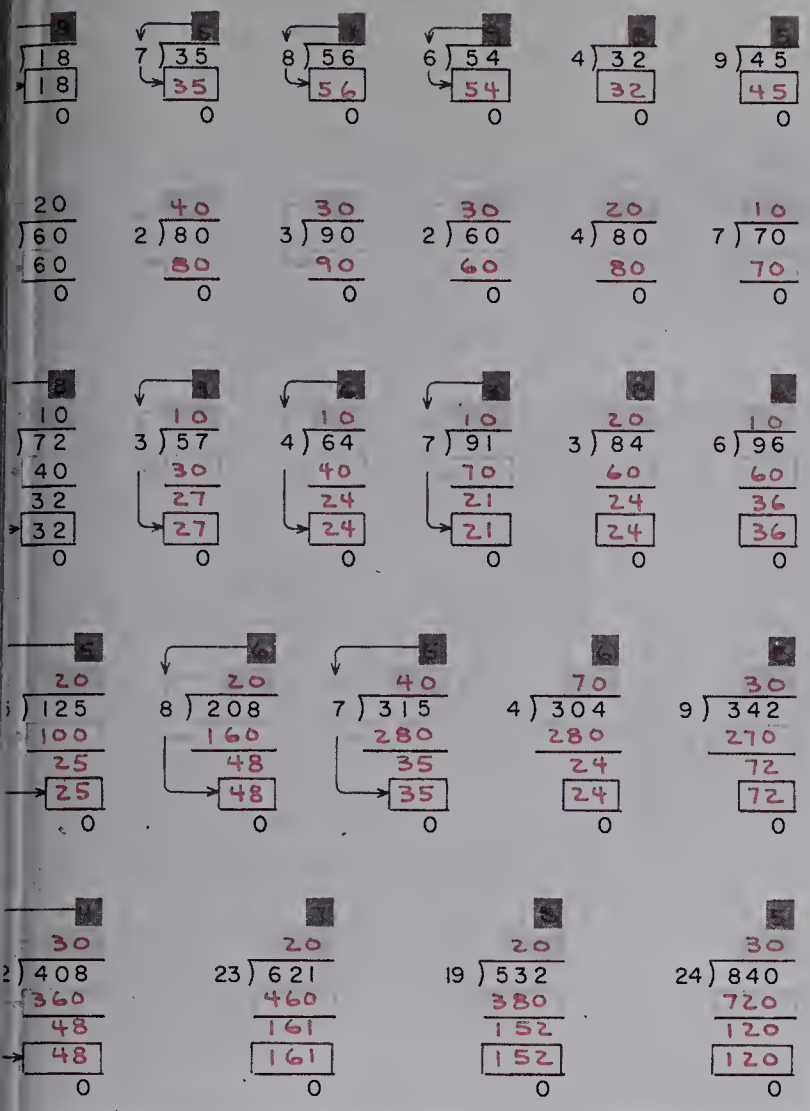
○ Reserved for multiples of 10

△ Reserved for numbers less then 10

Develop all these sketches on the board and let them remain to help introduce the ideas on *Work Text* pages 45, 46 and 47.

The activity on page 44 may be seen as leading toward asking the question: "Which sketch are we going to use to find the missing factor?"

is reserved for multiples of 10; ▪ is reserved for numbers less than 10.



The left-hand column lists the known factor in each case. The third, fourth and fifth columns from the left ask for a list of 10, 100 and 1000 times the known factor. Complete these three columns.

"Now, which sketch would we use to help see the problem in each example?"

Consider example 1: $8 \times$ ____ $= 768$ or $768 \div 8 =$ ____. We know that we cannot use sketch III, because $100 \times 8 = 800$ and that is more than 768. If we agree that sketch I is reserved for examples in which the missing factor is less than 10, then sketch I will not be adequate, because 768 is more than 10×8. This means that we must use sketch II—or, the unknown factor is more than 10 and less than 100 because 768 is more than 10×8 and less than 100×8—more than considered in sketch I and less than considered by sketch III.

If the unknown factor is more than 10 and less than 100, then it must be a two-digit number . . . which is the question asked in the fourth column from the right edge of the page.

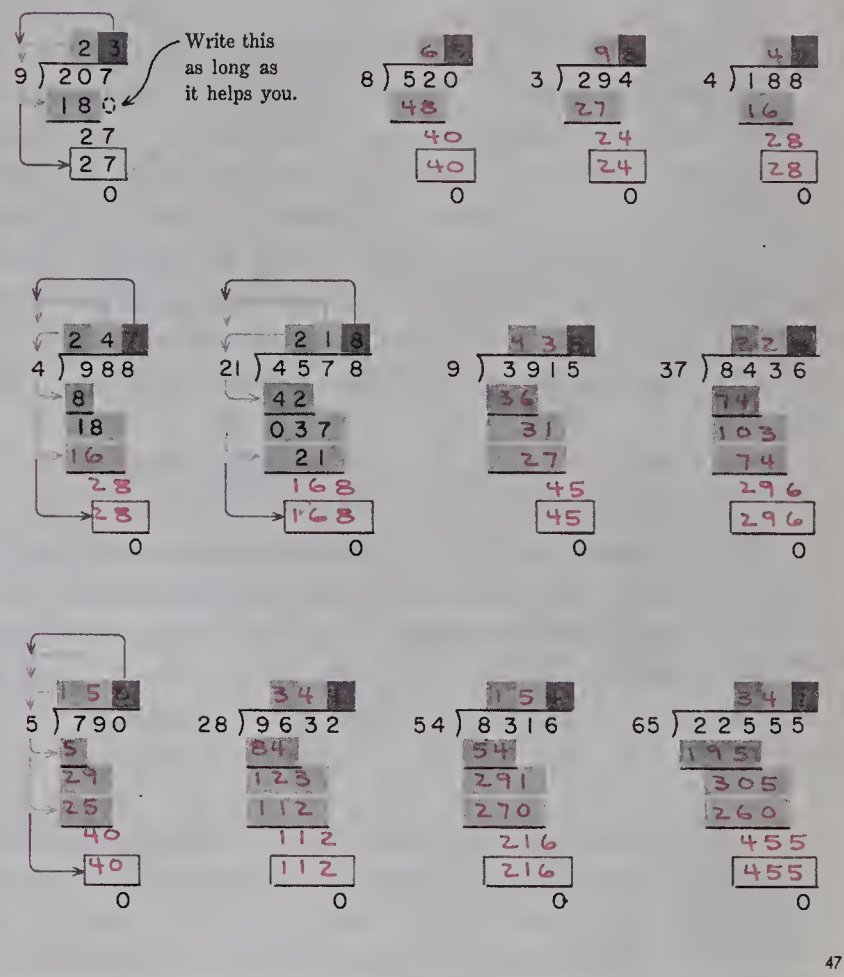
With some similar analysis of each example, this column ought to be completed before moving ahead.

47

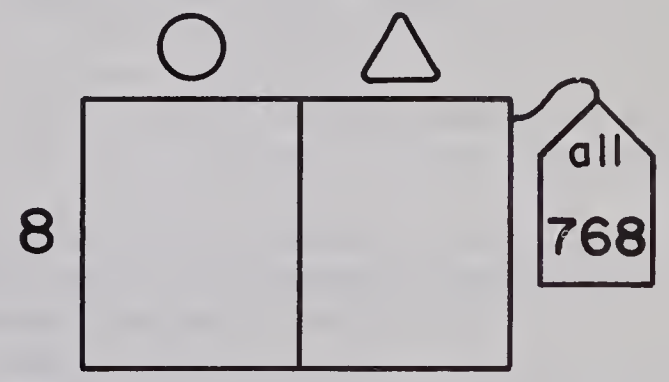
We agree that any number in the light green space ▪ is to be multiplied by 100.

And that any number written in the blue space ▪ is to be multiplied by 10.

Dark green ▪ is still reserved for numbers less than 10.



Now, in the sketch that we could use in each example, what will we write in the shape on the left? In example 1, the sketch is II. The given factor is 8; the product is 768; the unknown factor is a two-digit number.



What number that is a multiple of 10 will we write in the circle? We can experiment:

$8 \times 50 = 400$
$8 \times 60 = 480$
$8 \times 70 = 560$
$8 \times 80 = 640$
$8 \times 90 = 720$ ←
$8 \times 100 = 800$

So, we will write 90—because 8×90 is less than 768 and 8×100 is more than 768. This accounts for

8×90 or 720 of the 768, leaving 48 to be considered. This clearly points to 6 as the number to be written in the triangle. So, the unknown factor is $90 + 6$ or 96.

While this is a time-consuming kind of argument, it should be followed out as the class considers the first several examples, because many have been too willing to follow rules or procedures up to this point —and some have been "working division problems" without almost complete lack of understanding. Unless the argument is developed through several examples, some children will rely on old rules and not reason through to a conclusion.

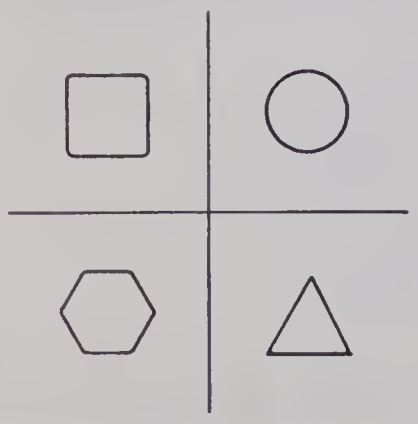
As you move on to the examples on *Work Text* pages 46 and 47, use sketches similar to those outlined above for typical problems. Ask the children to explain in terms of the appropriate sketch each step in these problems. They will see that in the examples on both pages they are, in effect, recording the kind of arguments used in the examples on page 45.

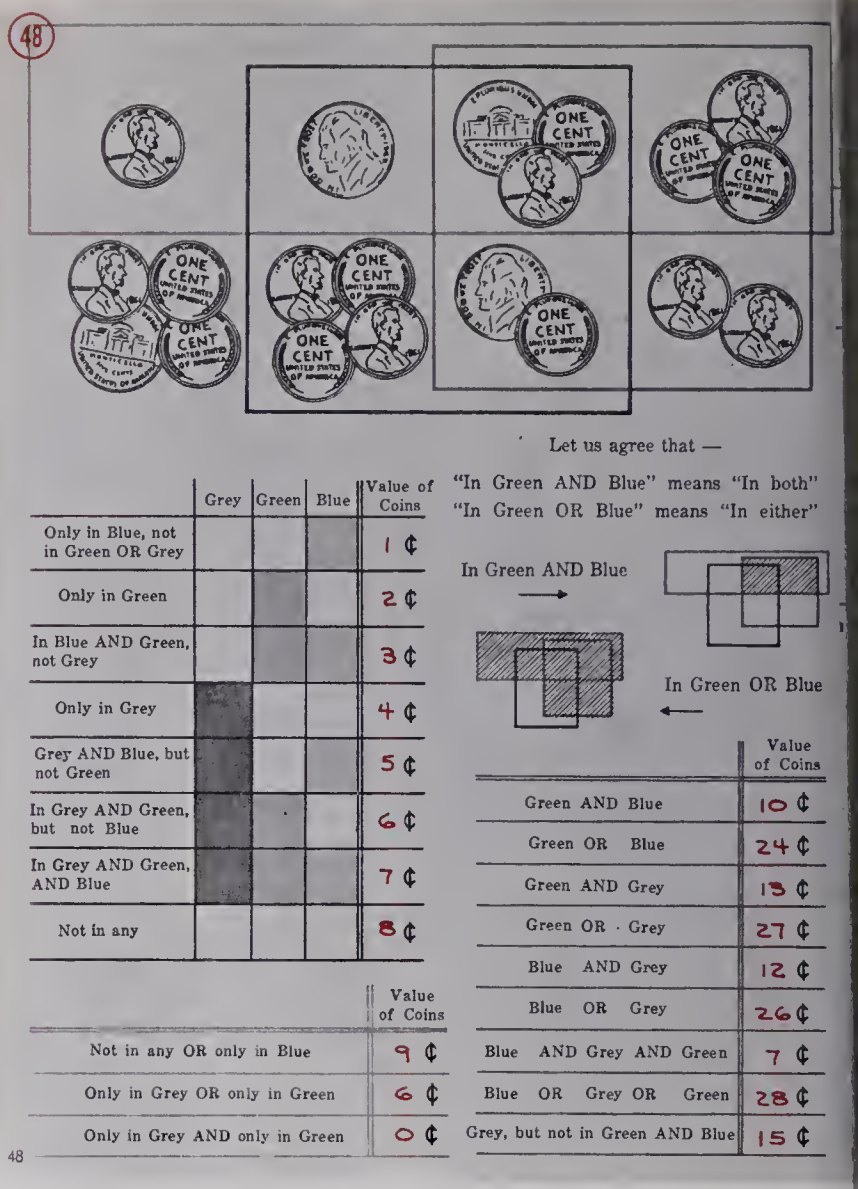
If some children have not met this scheme for finding the missing factor, supplementary examples would help develop a facility. They would need extra discussions to understand the shift in method of recording that takes place between pages 46 and 47—that, in effect, the means used on page 47 is simply a contraction of the form on page 46.

Throughout these pages, the main aim is to show that in division we are, in effect, *undoing* what multiplication *does*. The same diagram in each example serves equally well to support both the reasoning of multiplication and the reasoning of division.

Notes to accompany *Work Text* pages 48-49

The distinction between "AND" and "OR" as it is used in mathematics is investigated on *Work Text* pages 48 and 49. The distinction, of course, is purely a matter of definition—explaining an agreement of mathematicians in the use of these words. Without too much discussion, let's try some illustrations:



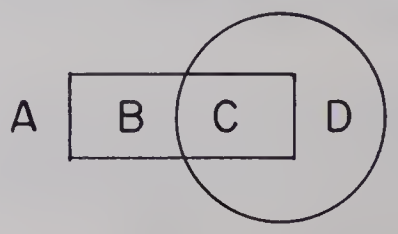

48
Let us agree that —
"In Green AND Blue" means "In both"
"In Green OR Blue" means "In either"
In Green AND Blue
In Green OR Blue
Grey
Green
Blue
Value of Coins
Only in Blue, not in Green OR Grey 1¢
Only in Green 2¢
In Blue AND Green, not Grey 3¢
Only in Grey 4¢
Grey AND Blue, but not Green 5¢
In Grey AND Green, but not Blue 6¢
In Grey AND Green, AND Blue 7¢
Not in any 8¢
Value of Coins
Not in any OR only in Blue 9¢
Only in Grey OR only in Green 6¢
Only in Grey AND only in Green 0¢
Value of Coins
Green AND Blue 10¢
Green OR Blue 24¢
Green AND Grey 13¢
Green OR Grey 27¢
Blue AND Grey 12¢
Blue OR Grey 26¢
Blue AND Grey AND Green 7¢
Blue OR Grey OR Green 28¢
Grey, but not in Green AND Blue 15¢
48

"How many shapes are above the horizontal line? . . . and what are they?" Two; the square and the circle.

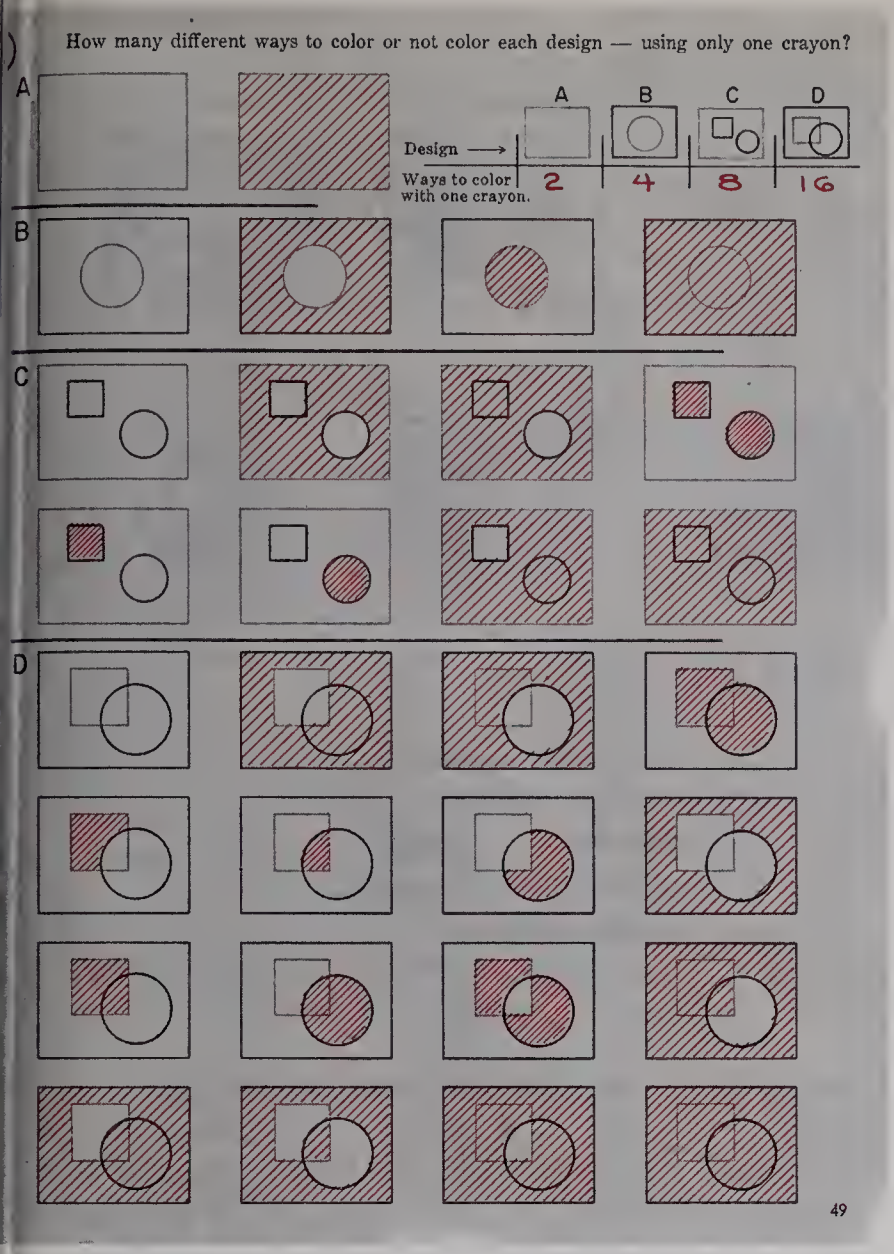
"How many shapes are to the right of the vertical line? . . . and what are they?" Two; the circle and the triangle.

"What shapes are both above the horizontal line *and* to the right of the vertical line?" Only the circle —it is the only shape that is both above *and* to the right.

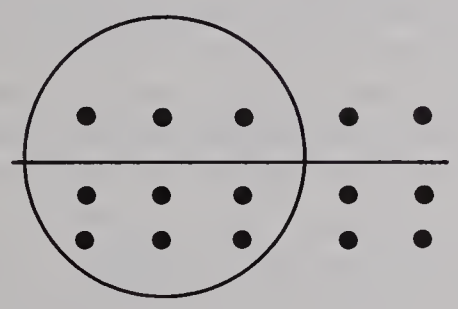
"What shapes are either above the horizontal line *or* to the right of the vertical line?" The square, circle and triangle.



"What letter or letters are inside the circle *and* inside the rectangle?" Only C.



"What letter or letters are inside the rectangle *or* circle?" B, C and D.



"How many dots are inside the circle *and* above the line?" (3.)

"How many dots are inside the circle *and* below the line?" (6.)

"How many dots are outside the circle *and* above the line?" (2.)

"How many dots are outside the circle *and* below the line?" (4.) Ask the same questions, using "or" instead of "and."

"How many dots are inside the circle *or* above the line?" (11.)

"How many dots are inside the circle *or* below the line?" (13.)

"How many dots are outside the circle *or* above the line?" (9.)

"How many dots are outside the circle *or* below the line?" (12.)

To explain the idea further, imagine each dot (or letter or shape) as the answer to each of the two questions implicit in the first question above:

"Are you inside the circle?"

"Are you above the line?"

How many could answer "yes" to *both* questions . . . the first *and* second questions? (Only the three that are inside the circle *and* at the same time above the line.)

How many could answer "yes" to at least one of the questions? . . . one question *or* the other? (That some could answer "yes" twice doesn't disqualify them.)

More illustrations

This idea is mathematically important—we must have a clear understanding of what we agree that "AND" means and what "OR" means. Keep finding more and more illustrations until the definition is clear. Here is another tack:

"How many whole numbers are less than 10 *and* are (also) odd numbers?" 1, 3, 5, 7, 9—five numbers.

"How many whole numbers are less than 10 *or* are odd numbers?" 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, etc.—the number of candidates is unlimited.

"How many whole numbers (excluding 0) less than 25 are exactly divisible by 4?" 4, 8, 12, 16, 20, 24—six numbers.

"How many whole numbers (excluding 0) less than 25 are exactly divisible by 3?" 3, 6, 9, 12, 15, 18, 21, 24—eight numbers.

"How many are divisible by (both) 3 *and* 4?" 12 and 24—two numbers.

"How many are divisible by (either) 3 *or* 4?" 3, 4, 6, 8, 9, 12, 15, 16, 18, 20, 21, 24—twelve numbers.

"How many 2-digit numbers are exactly divisible by 9?" 18, 27, 36, 45, 54, 63, 72, 81, 90, 99—ten numbers.

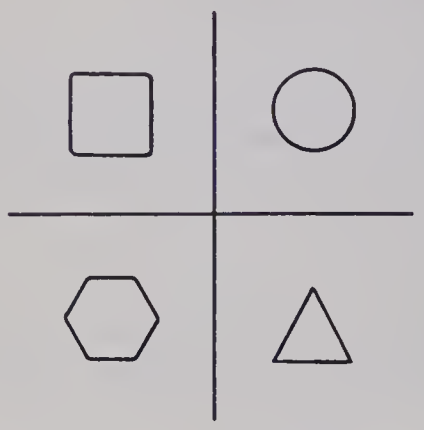
"How many 2-digit numbers are exactly divisible by 11?" 11, 22, 33, 44, 55, 66, 77, 88, 99—nine numbers.

"How many 2-digit numbers are exactly divisible by 9 *and* 11?" 99—only one.

"How many 2-digit numbers are exactly divisible by 9 *or* 11?" 11, 18, 22, 27, 33, 36, 44, 45, 54, 55, 63, 66, 72, 77, 81, 88, 90, 99—eighteen numbers.

A third definition

Now, let's add another word—"NOT."



"What shapes are *not* above the line?"2—hexagon and triangle.

"What shapes are to the right of the line?" 2—circle and triangle.

"How many are to the right of the line *and* not above the line?" Only the triangle could answer "yes" to both these questions . . . so, only one shape is both to the right *and* not above.

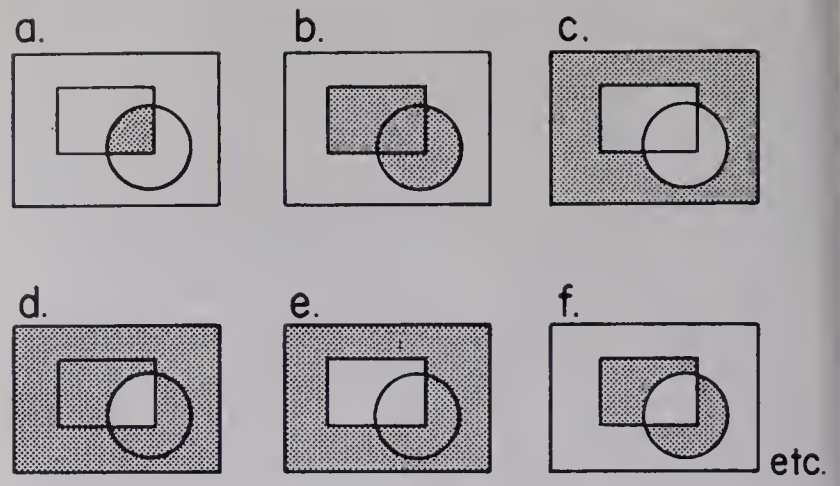
With the introduction of this third word—not—the questions we can ask are many.

What has happened is that we now have a neat bit of precise language we will use more and more as we move on in mathematics. Consider its use to talk about the coloring problems on page 49.

In the second series there are four possibilities:

1) not in the circle *or* not outside the circle.
2) outside the circle *and* not inside the circle. (We could also use "or.")
3) inside the circle *or* not outside the circle.
4) outside the circle or inside the circle.

Let's consider the last diagram. Some possibilities are indicated:



What areas are shaded?

a. In circle *and* square.
b. In circle *or* square.
c. Not in circle *or* square.
d. Not in circle *and* square.
e. Not in square.
f. In circle or square, but not in circle *and* square.

Some pupils will have encountered this idea of "AND" and "OR" as it is defined in mathematics. All will encounter it again and again. Mastery is, obviously, not our concern. However, do not feel that you are spending too much time if you linger over one or both of these pages—or on your own version of the idea.

You may recognize the idea of "AND" as the same as the idea in set language that employs the term "intersections" of sets; and the idea of "OR" as the idea in set language that employs the term "union" of sets. Since our primary concern is on concepts rather than terms, we suggest limiting the language to that used in the *Work Text*. However, if you want to use "set language," and feel comfortable with it, or wish to experiment, you should, of course.

Notes to accompany *Work Text* page 50

If Jim had a 1-ounce and a 2-ounce weight, how could he weigh 2 ounces of sugar on the balance? . . . 1 ounce? . . . 3 ounces?

Are there other amounts that he could weigh with only a 1- and a 2-ounce weight? Let's include "no" ounces (0 ounces). So, the weights he could *measure* are:

ounces—0, 1, 2, 3 (*four* measurements) with 0, 1- and 2-ounce weights.

Suppose we add a 4-ounce weight to the supply. How many weights can he make now?

ounces—0, 1, 2, 3, 4, 5, 6, 7
(*eight* measurements)
with 0-, 1-, 2- and 4-ounce weights.

0	4
1	4 + 1
2	4 + 2
2 + 1	4 + 2 + 1

Now, add an 8-ounce weight to the 4-ounce, 2-ounce, 1-ounce and 0-ounce weights:

0	4	8	8 + 4
1	4 + 1	8 + 1	8 + 4 + 1
2	4 + 2	8 + 2	8 + 4 + 2
2 + 1	4 + 2 + 1	8 + 2 + 1	8 + 4 + 2 + 1

(16 measurements)

What weight do you think he would want to add next? (16 ounces.) Then, how many measurements could he make? All he made before (16), plus the same with the 16-ounce weight added to each . . . 32 in all. What is the largest measurement he could make? 31 ounces—16 + 8 + 4 + 2 + 1.

Lights

In how many ways could you signal with one green light that could be turned on and off? Two—the light *on* and the light *off*.

Suppose you had 1 green and 1 blue light: both off; green on; blue on; both on.

Suppose you had 1 green, 1 blue and 1 yellow light?—all three off; green on; blue on; green and blue on; yellow on; yellow and green on; yellow and blue on; yellow and blue and green.

Suppose you had 1 green, 1 blue, 1 yellow, and 1 red light?—all eight combinations above, and all those combinations plus the red light.

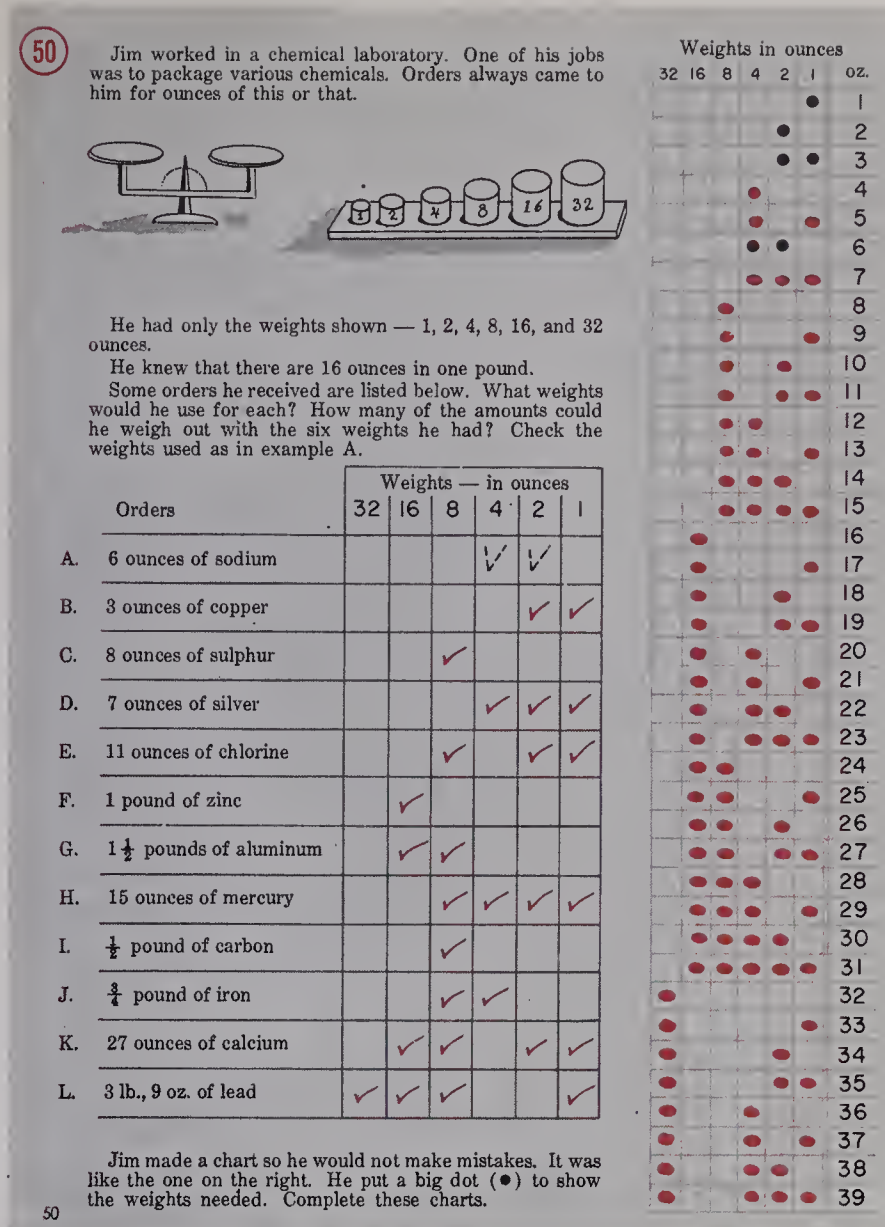
Suppose you add a white light?—all 16 above, plus all 16 above and the white light.

Coins

With only a penny, how many amounts of money can you make?—0¢ and 1¢.

With a nickel and a penny?—0¢ and 1¢; 5¢ and 6¢ (penny and nickel).

With a dime and a nickel and a penny?—



50

Jim worked in a chemical laboratory. One of his jobs was to package various chemicals. Orders always came to him for ounces of this or that.

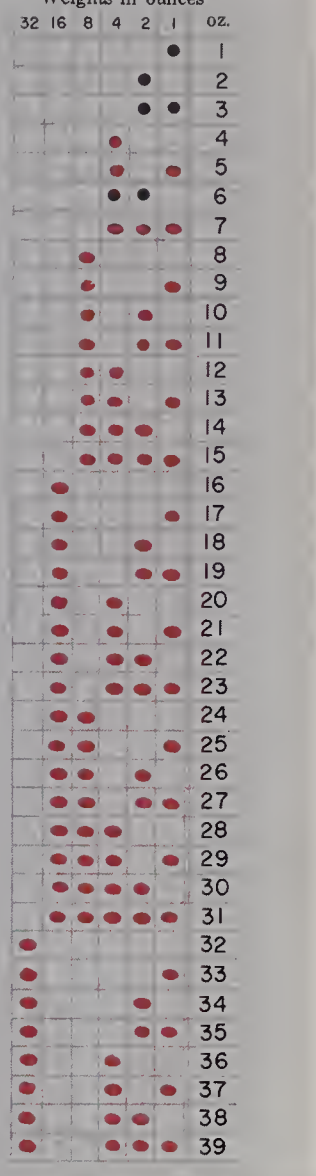
He had only the weights shown — 1, 2, 4, 8, 16, and 32 ounces.

He knew that there are 16 ounces in one pound.

Some orders he received are listed below. What weights would he use for each? How many of the amounts could he weigh out with the six weights he had? Check the weights used as in example A.

	Orders	Weights — in ounces 32	16	8	4	2	1
A.	6 ounces of sodium				✓	✓	
B.	3 ounces of copper					✓	✓
C.	8 ounces of sulphur			✓			
D.	7 ounces of silver				✓	✓	✓
E.	11 ounces of chlorine			✓		✓	✓
F.	1 pound of zinc		✓				
G.	1½ pounds of aluminum		✓	✓			
H.	15 ounces of mercury			✓	✓	✓	✓
I.	½ pound of carbon			✓			
J.	¾ pound of iron			✓	✓		
K.	27 ounces of calcium		✓	✓		✓	✓
L.	3 lb., 9 oz. of lead	✓	✓	✓			✓

Jim made a chart so he would not make mistakes. It was like the one on the right. He put a big dot (•) to show the weights needed. Complete these charts.



	Dimes	Nickels	Pennies
0¢	0	0	0
1¢	0	0	1
5¢	0	1	0
6¢	0	1	1
10¢	1	0	0
11¢	1	0	1
15¢	1	1	0
16¢	1	1	1

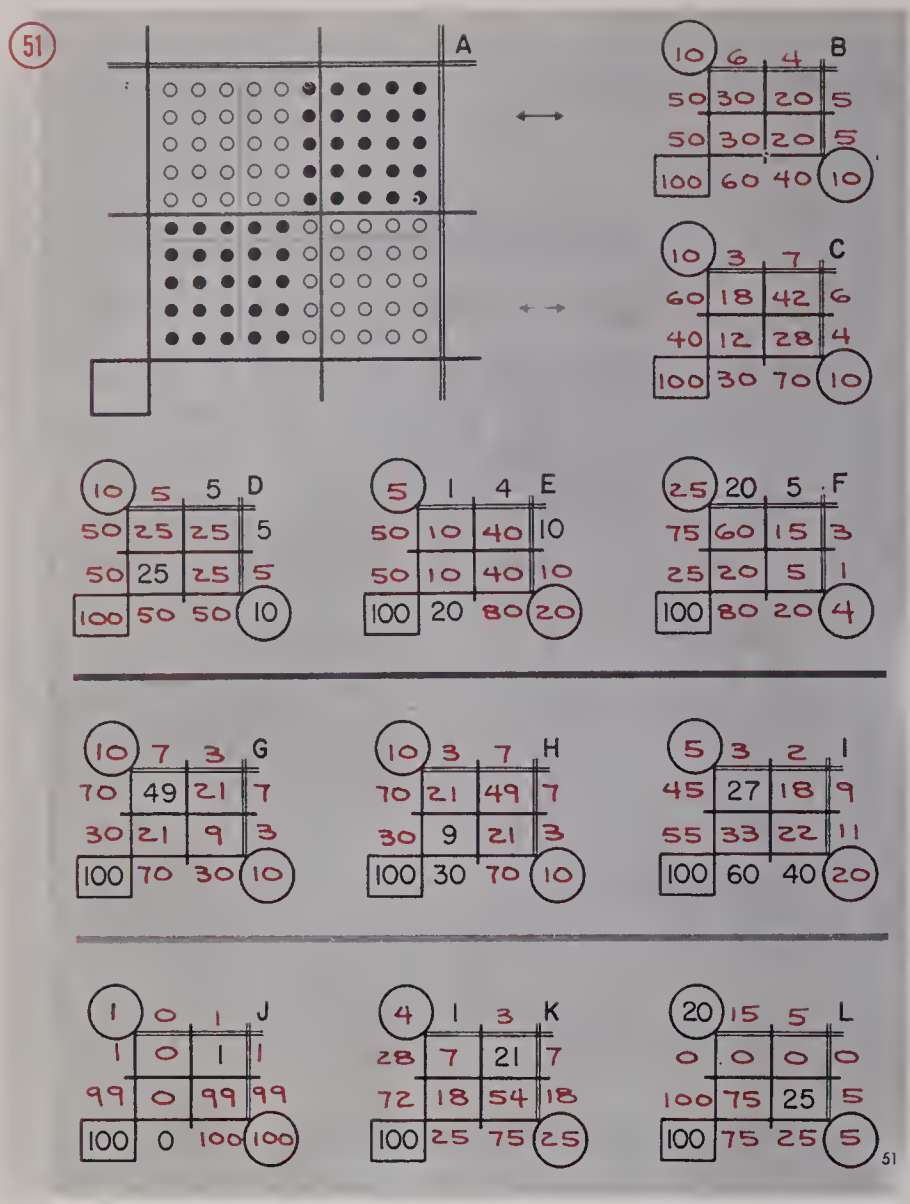
With a quarter, a dime, a nickel and a penny? (16 different amounts.)

With a half-dollar and the 4 coins above? (32 different amounts.)

Now turn to *Work Text* page 50, and this pattern will be put to use. (Some teachers want to refer to the "binary system" used by computers.)

Notes to accompany *Work Text* page 51

Children need to be reminded often that multiplication facts reflect patterns in the world of things.



Each of the basic multiplication facts has many representations, one of which is a "row-column" model: 3×4 expresses the idea of 3 rows and 4 columns, or 4 rows and 3 columns.

Work Text page 51 is a brief analysis of 100 as a product of two factors. The diagram shows a row-column interpretation of $10 \times 10 = 100$. This illustration can be "regrouped" in many ways. Those dots above the green line are an example of 5 rows and 10 columns—50 dots. Below the green lines—5 rows and 10 columns or 50 dots. This suggests that $5 \times 10 + 5 \times 10 = (5 + 5)\ 10 = 10 \times 10 = 100$.

The vertical green line separates the dots into two groups—10 rows and 6 columns, and 10 rows and 4 columns. This suggests that:

$$10 \times 6 + 10 \times 4 = 10 \times (6 + 4) =$$
$$10 \times 10 = 100, \text{ or}$$
$$6 \times 10 + 4 \times 10 = (6 + 4) \times 10 =$$
$$10 \times 10 = 100$$

The blue lines create another "regrouping" of a similar nature.

We can also look at another kind of "regrouping" of the 100 dots. "How many dots are above a green line *and* to the left of a green line?" 5 rows and 6 columns, or 36.

"How many dots are above a green line *and* to the right of a green line?" 5 rows and 4 columns, or 24. $(5 \times 6 + 5 \times 4 = 5 \times (6 + 4) = 5 \times 10 = 50)$

"How many are below a green line *and* to the left? . . . below and to the right?"

The answers to these questions is recorded in the form indicated by the green arrow. Results of a similar questioning about the regrouping indicated by the blue lines are recorded in the form labelled *C*.

Other regroupings of 100 dots are suggested by the incomplete Cross Number Puzzle forms D through L. If more of this kind of review seems necessary for all or some of the class, refer to the *Work Forms*. There is a series (C4 through C10) that analyzes the basic facts in which factors are 2 through 10. Refer also to C14 which is a full page of record blanks such as used on page 51.

Remember that these facts are an essential part of the vocabulary of arithmetic and must be known automatically by pupils!

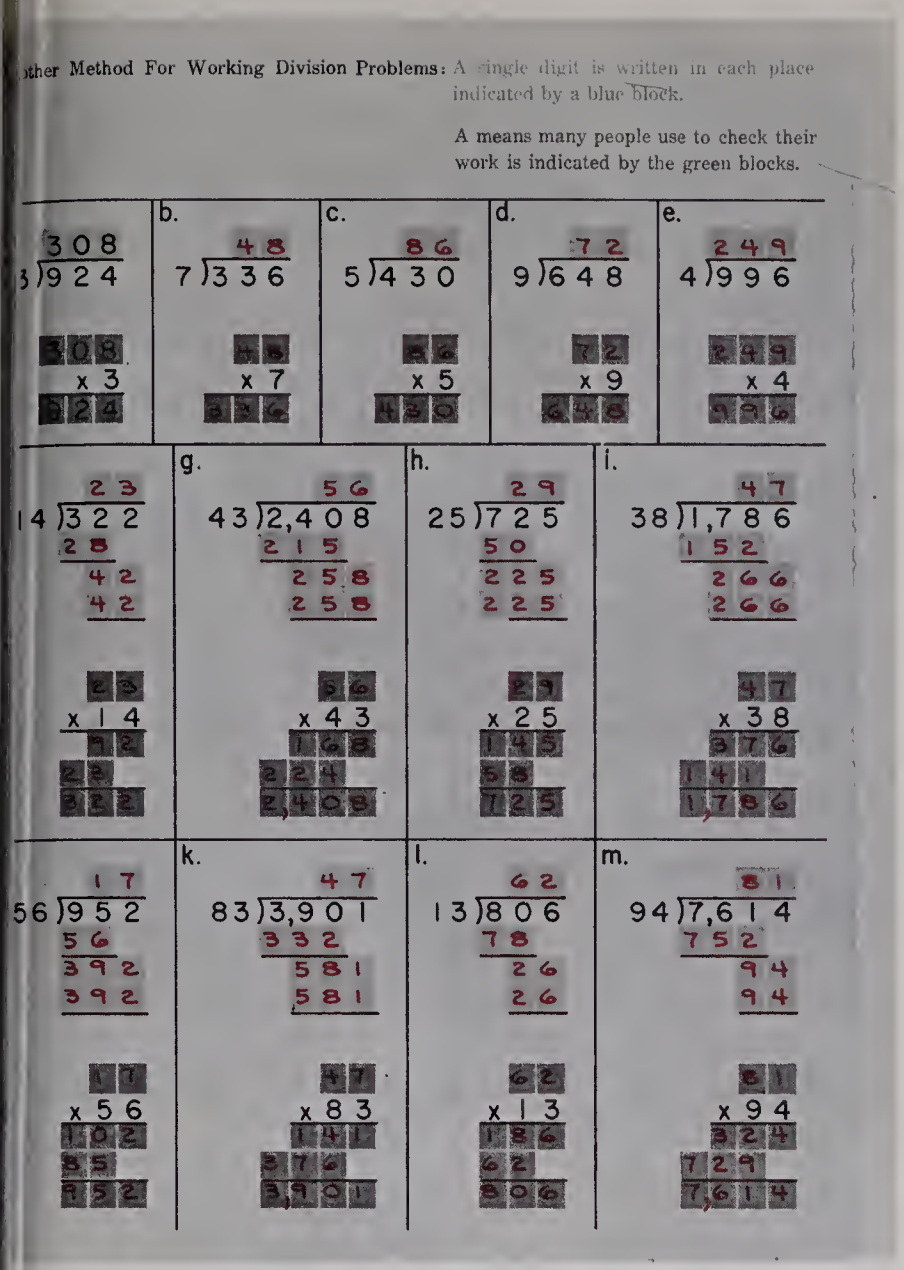
Notes to accompany *Work Text* pages 52-53

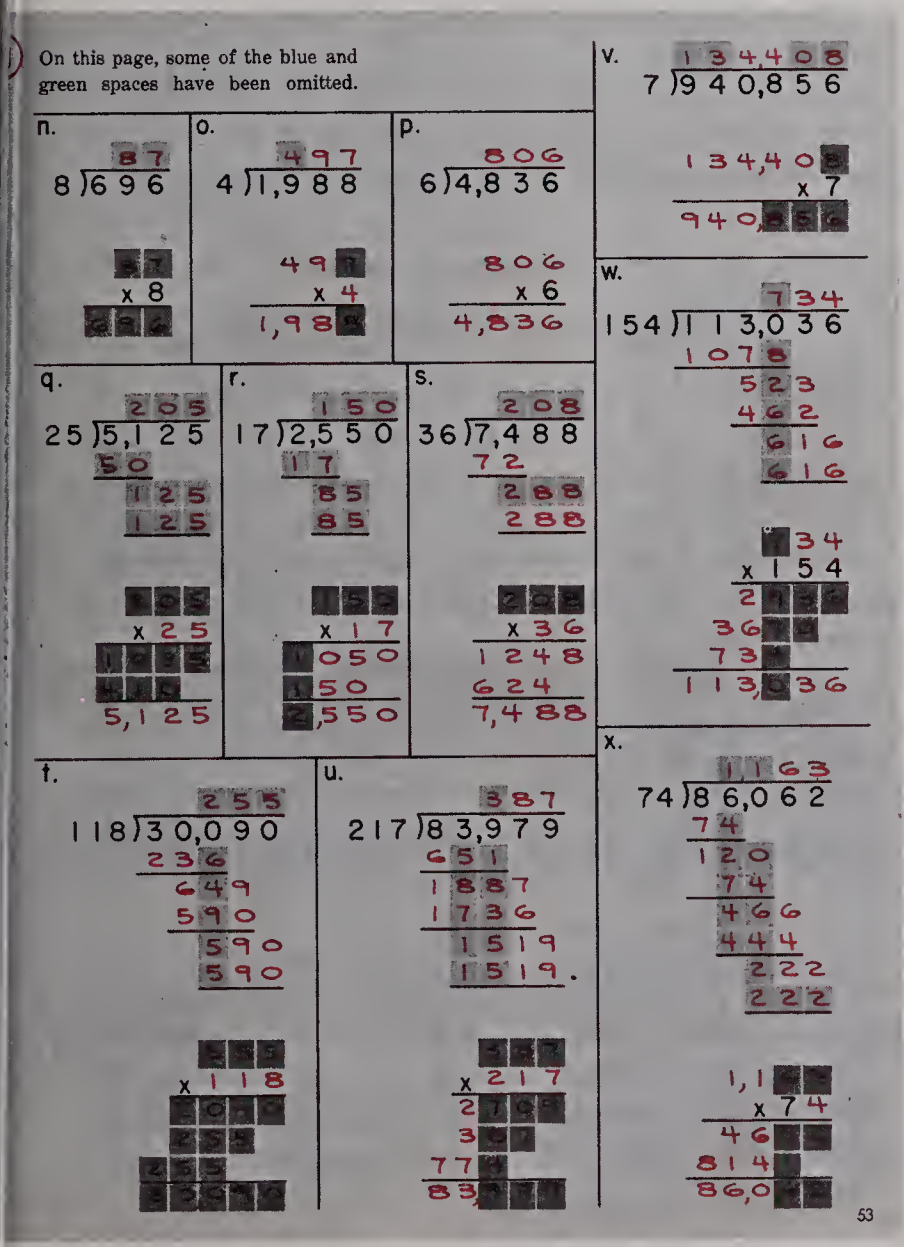
Consider these two pages for independent work. Simply ask children to work through them without any prior discussion and with a minimum of help.

For many, this will be very simple and familiar work in computation. However, children, even at this age, like to fill in blanks—and this points to one of the purposes behind these pages.

Children develop habits of careless arrangement of their work. This can lead to incorrect computations and ought to be discouraged. In pages arranged in this fashion, they are almost *forced* to work in an orderly way—in an arrangement that contributes to understanding and accuracy.

Each of the 24 examples repeats another important fact—that division and multiplication are clearly related: in division we have a factor and product and are looking for a missing factor; in multiplication we have two factors and are looking for the missing product.


ther Method For Working Division Problems: A single digit is written in each place indicated by a blue block.
A means many people use to check their work is indicated by the green blocks.



Notes to accompany *Work Text* pages 54-55

Emphasis in the next several pages is on developing facility in working with fractions and understanding the several concepts fractions express. The work moves back and forth between a very formal approach (using the device of an Alec Marson machine) and a very intuitive treatment.

We will need to reach an agreement as we jump from the concrete to the abstract and back to the world of things. In the way we use the pattern of fractions in everyday life, the word "of" implies the idea expressed by the multiplication sign in arithmetic:

"One-half of six is three"—$\frac{1}{2} \times 6 = 3$

"Give me 3 of those quarters"—$3 \times \frac{1}{4} = \frac{3}{4}$

"One-third of a half a pie is one-sixth of a pie"—$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

Let's make a very specific point of this with children: "of" in talking about events can be translated into "times" ($\times$) when we express patterns in arithmetic.

Until we meet fractions, we do not have this problem—because the language of the real world and arithmetic are closely parallel.

"I weigh 3 *times* as much as my baby brother, who weighs 25 pounds." $3 \times 25 = 75$.

"My weight is $\frac{1}{2}$ *of* my father's weight. He weighs 188 pounds." $\frac{1}{2} \times 188 = 94$.

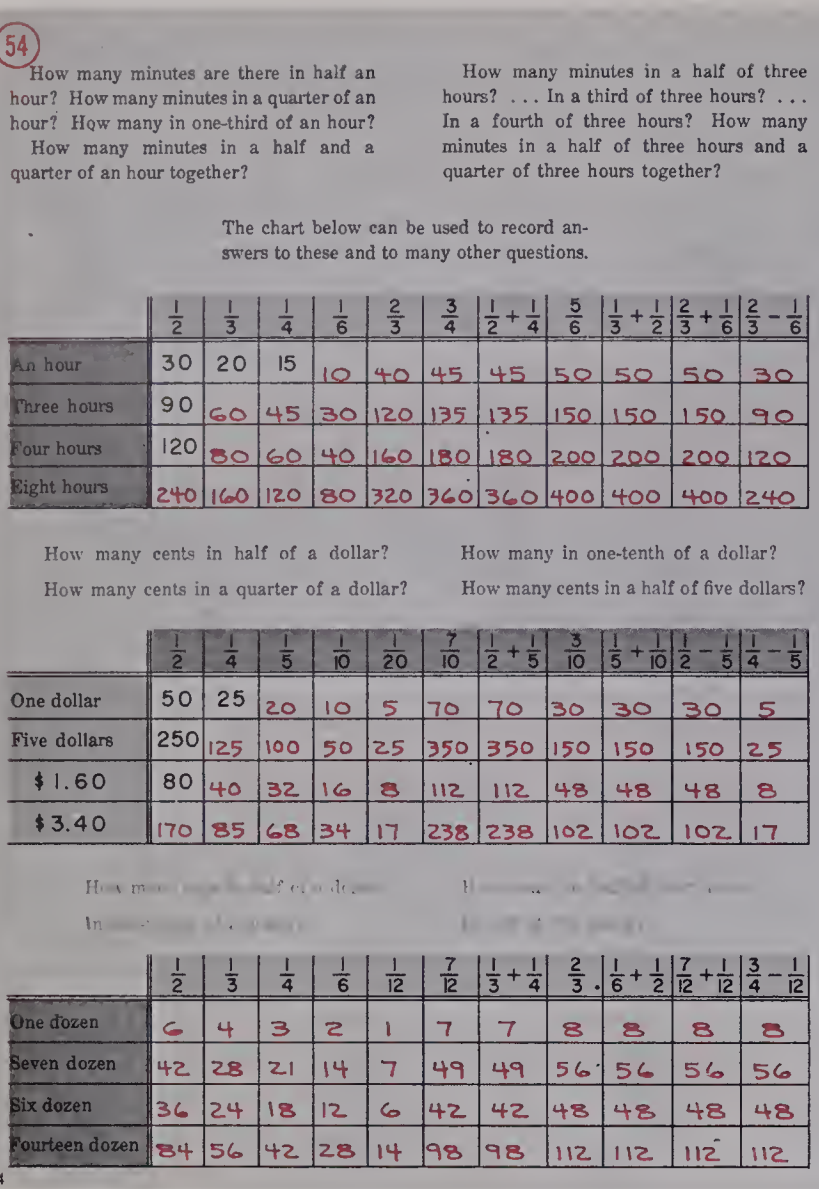
Speak of this fact that "of" as it is used in talking about events and "times" as it is used in arithmetic, point to the same pattern. Throughout *Work Text* pages 55 and 56 you will have countless opportunity for this. The first group of questions are in the form of "$\frac{1}{2}$ of 24" which ought to be written: $\frac{1}{2} \times 24 = 12$. But when we use this as a headline, the "$\times$" leads us often to use "of" in the story.

"I have 24 books, and I have read one-half *of* them."

"One-half *of* one day is 12 hours."

"We have used half *of* the two dozen eggs mother bought yesterday."

In finding answers to the questions asked, in effect, by the first two charts, we hope the argu-



54

How many minutes are there in half an hour? How many minutes in a quarter of an hour? How many in one-third of an hour?

How many minutes in a half and a quarter of an hour together?

How many minutes in a half of three hours? . . . In a third of three hours? . . . In a fourth of three hours? How many minutes in a half of three hours and a quarter of three hours together?

The chart below can be used to record answers to these and to many other questions.

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{2}+\frac{1}{4}$	$\frac{5}{6}$	$\frac{1}{3}+\frac{1}{2}$	$\frac{2}{3}+\frac{1}{6}$	$\frac{2}{3}-\frac{1}{6}$
An hour	30	20	15	10	40	45	45	50	50	50	30
Three hours	90	60	45	30	120	135	135	150	150	150	90
Four hours	120	80	60	40	160	180	180	200	200	200	120
Eight hours	240	160	120	80	320	360	360	400	400	400	240

How many cents in half of a dollar? How many in one-tenth of a dollar?

How many cents in a quarter of a dollar? How many cents in a half of five dollars?

	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{7}{10}$	$\frac{1}{2}+\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{5}+\frac{1}{10}$	$\frac{1}{2}-\frac{1}{5}$	$\frac{1}{4}-\frac{1}{5}$
One dollar	50	25	20	10	5	70	70	30	30	30	5
Five dollars	250	125	100	50	25	350	350	150	150	150	25
$1.60	80	40	32	16	8	112	112	48	48	48	8
$3.40	170	85	68	34	17	238	238	102	102	102	17

How many [illegible] dozen? How many [illegible]

In [illegible]

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{7}{12}$	$\frac{1}{3}+\frac{1}{4}$	$\frac{2}{3}$	$\frac{1}{6}+\frac{1}{2}$	$\frac{7}{12}+\frac{1}{12}$	$\frac{3}{4}-\frac{1}{12}$
One dozen	6	4	3	2	1	7	7	8	8	8	8
Seven dozen	42	28	21	14	7	49	49	56	56	56	56
Six dozen	36	24	18	12	6	42	42	48	48	48	48
Fourteen dozen	84	56	42	28	14	98	98	112	112	112	112

55

How many is one-half of 24? How many is one-fourth of 24? . . . One-sixth of 24?

How many is one-half of 120? How many is two-thirds of 120?

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{3}{8}+\frac{1}{8}$	$\frac{1}{6}+\frac{1}{3}$	$\frac{5}{6}$	$\frac{1}{3}+\frac{1}{2}$	$\frac{1}{2}$
24	12	8	6	4	3	12	12	20	20	4
120	60	40	30	20	15	60	60	100	100	20
216	108	72	54	36	27	108	108	180	180	36
168	84	56	42	28	21	84	84	140	140	28
336	168	112	84	56	42	168	168	280	280	56

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{3}-\frac{1}{6}$	$\frac{1}{9}+\frac{2}{9}$	$\frac{1}{6}+\frac{1}{3}$	$\frac{1}{3}-\frac{1}{9}$	$\frac{1}{2}-\frac{1}{6}$	$\frac{5}{9}-$
36	18	12	6	4	6	12	18	8	12	12
180	90	60	30	20	30	60	90	40	60	60
144	72	48	24	16	24	48	72	32	48	48
252	126	84	42	28	42	84	126	56	84	84
324	162	108	54	36	54	108	162	72	108	108

Results in the charts on these two pages suggest that the following might be true statements.

$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ $\quad \frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ $\quad \frac{2}{3}+\frac{1}{6}=\frac{5}{6}$ $\quad \frac{2}{3}-\frac{1}{6}=$

$\frac{1}{2}+\frac{1}{5}=\frac{7}{10}$ $\quad \frac{1}{5}+\frac{1}{10}=\frac{3}{10}$ $\quad \frac{1}{2}-\frac{1}{5}=\frac{3}{10}$ $\quad \frac{1}{4}-\frac{1}{5}=$

$\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$ $\quad \frac{1}{6}+\frac{1}{2}=\frac{2}{3}$ $\quad \frac{7}{12}+\frac{1}{12}=\frac{2}{3}$ $\quad \frac{3}{4}-\frac{1}{12}=$

$\frac{3}{8}+\frac{1}{8}=\frac{1}{2}$ $\quad \frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ $\quad \frac{1}{6}+\frac{1}{3}=\frac{1}{2}$ $\quad \frac{5}{9}-\frac{2}{9}=$

Do you believe that these statements are true?

ments go in this way: The first line of the chart on page 54 asks such questions as:

"How many minutes in one-half of an hour?" (30 minutes)

"How many minutes in one-quarter of an hour?" (15 minutes)

"How many minutes in three-quarters of an hour?" (45 minutes)

"How many minutes in a half-hour plus a quarter of an hour?" (45 minutes)

This suggests the idea that $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Since $\frac{1}{6}$ of an hour is 10 minutes, how much is $\frac{5}{6}$ of an hour? (50 minutes)

Since $\frac{1}{3}$ of an hour is 20 minutes and $\frac{1}{2}$ of an hour is 30 minutes, how much is that altogether? (50 minutes) This suggests that $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

Already you have come up with a conclusion that is far from obvious to anyone without computation of some kind or mentally constructing a rather complicated picture.

"Does this result make sense?" Yes, in terms of parts of an hour.

"Is $\frac{1}{3}$ of a dozen of eggs plus $\frac{1}{2}$ of a dozen the same number of eggs as $\frac{5}{6}$ of a dozen eggs?"

Write out such statements, including the arithmetic implied:

$$(\tfrac{1}{3} \times 12) + (\tfrac{1}{2} \times 12) = \tfrac{5}{6} \times 12$$
$$(\tfrac{1}{3} + \tfrac{1}{2}) \times 12 = (\tfrac{5}{6}) \times 12$$
$$\tfrac{1}{3} + \tfrac{1}{2} = \tfrac{5}{6}$$

Now, this should have nothing to do with manipulation—rather, each expression makes sense in terms of eggs:

"A third of a dozen plus a half-dozen is the same as five-sixths of a dozen."

"A third plus a half a dozen is five-sixths of a dozen."

"This suggests that $\frac{1}{3} + \frac{1}{2}$ is the same number as $\frac{5}{6}$."

Some children are already adept at using "least common denominators" and arguing that: $\frac{1}{3} + \frac{1}{2}$

$= \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$. And it is well that they, too, be required to think through such examples in terms of what happens in the world of eggs and hours and miles and money.

In discussing the statements at the bottom of page 55, point to the concluding question: "Do you believe that these statements are true?" The answers that ought to be encouraged are in terms of patterns from experience—not as a consequence of arithmetic manipulations.

"I believe that $\frac{5}{9} - \frac{2}{9} = \frac{1}{3}$ because, if we had 5 players out of the 9 we need for a full baseball team and 2 players went home, we would have 3 players left—which is only $\frac{1}{3}$ of a full team."

Keep directing attention on these two pages to the world of "things"—particularly because the next four pages will take a very different approach (in the world of Alec's machines).

Notes to accompany *Work Text* pages 56-57

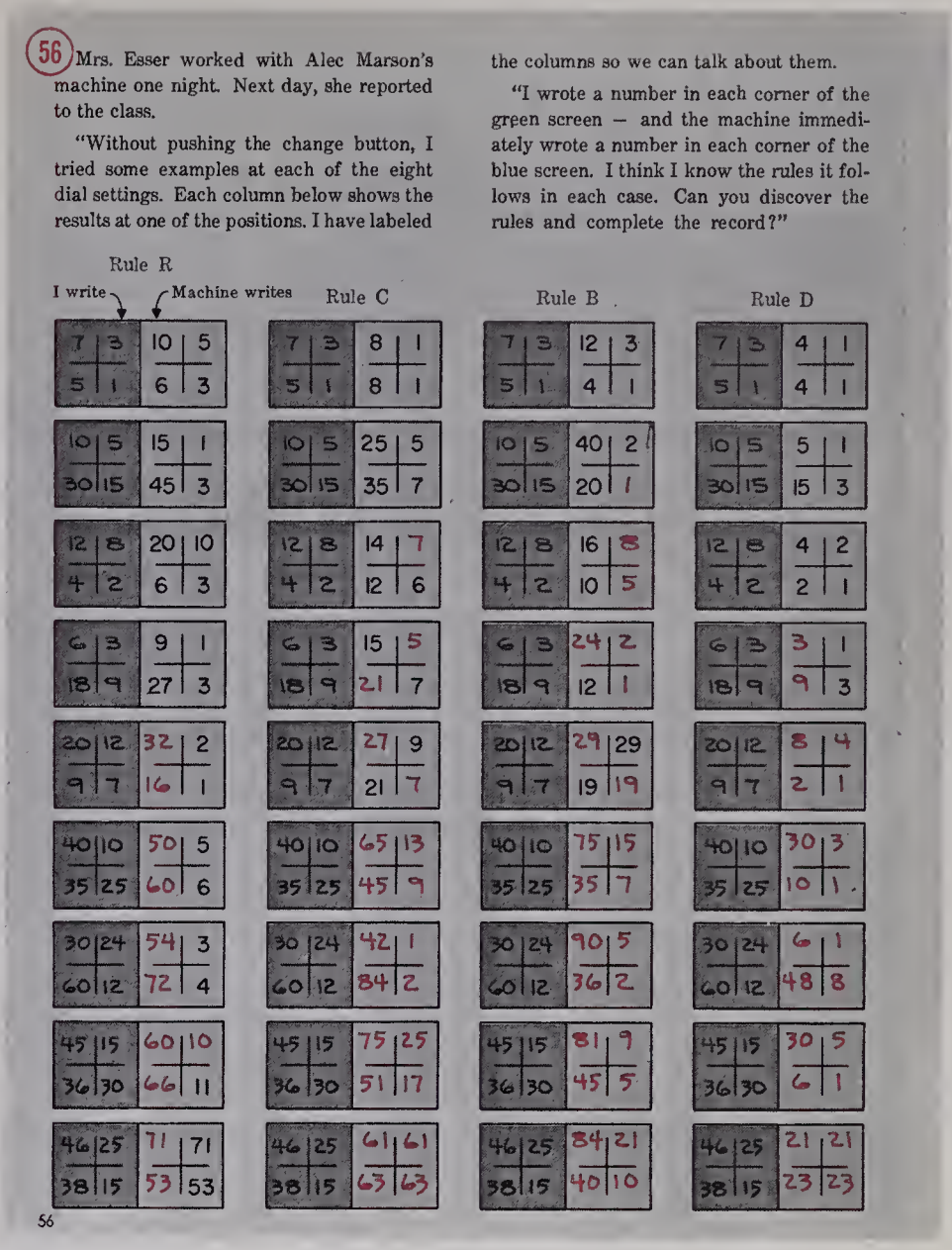
Children who have learned to search for pat rules that apply will be most uncomfortable as they tackle *Work Text* page 56.

A game or two of "What's My Rule?" might best set the mood for the work at hand. Alec's machines always follow some rule—and they are not always familiar rules.

A →	
3,4	7,1
15,9	24,6
0,8	8,8
4,4	8,0
7,5	__,__

B →	
8,4	9,6
1,7	2,9
3,3	4,5
10,9	11,11
6,4	__,__

"What's the rule in chart A?" By what rule does 3,4 "go to" 7,1? . . . 9,15 "go to" 24,6 . . . etc. (Secret for A: any pair of numbers "goes to" another pair, the first of which is the sum and the other the difference of the first pair:


56
Mrs. Esser worked with Alec Marson's machine one night. Next day, she reported to the class.
"Without pushing the change button, I tried some examples at each of the eight dial settings. Each column below shows the results at one of the positions. I have labeled the columns so we can talk about them.
"I wrote a number in each corner of the green screen – and the machine immediately wrote a number in each corner of the blue screen. I think I know the rules it follows in each case. Can you discover the rules and complete the record?"
Rule R
I write
Machine writes
Rule C
Rule B
Rule D
56

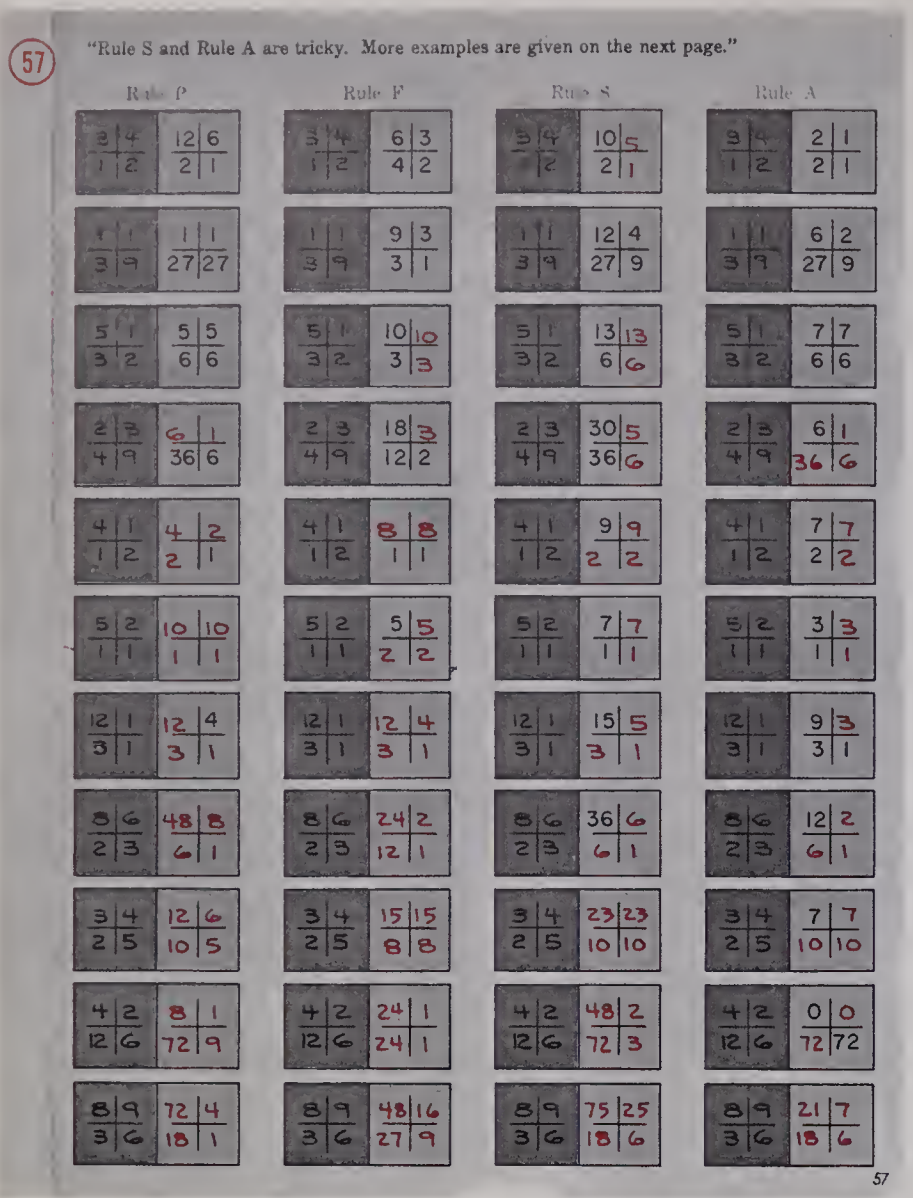
3,4 → (3 + 4, (4 − 3) or 7,1
15,9 → (15 + 9), (15 − 9) or 24,6
0,8 → (0 + 8), (8 − 0) or 8,8

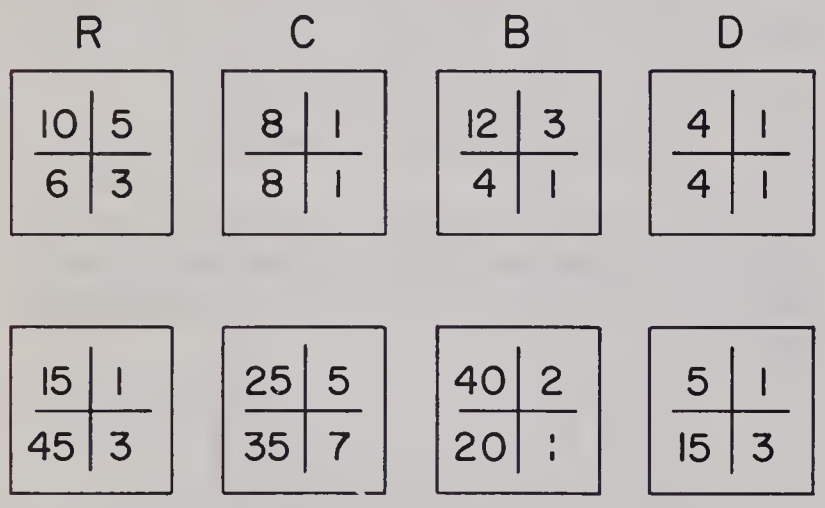
(Secret for B: each pair "goes to" another pair in which the first member is 1 more than and the second is 2 more than the corresponding member of the first pair.

8,4 → (8 + 1), (4 + 2) or 9,6
1,7 → (1 + 1), (7 + 2) or 2,9
3,3 → (3 + 1), (3 + 2) or 4,5
10,9 → (10 + 1), (9 + 2) or 11,11

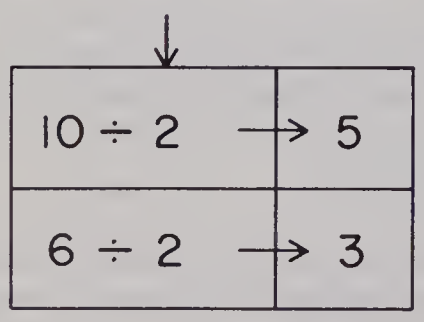
Such introductory activity anticipates the rules that Alec's machine seems to be following on pages 56 and 57. These rules are labeled R, C, B, D, P, F, S and A. (The first four on page 56 are pure games, but the four on page 57 bear a close relationship to finding a product (P), finding a factor (F), finding a sum (S), and finding an addend (A) when a pair of fractions is involved. However, let them all be games at this point.)

One of the first things we notice in all examples given is a relationship among the four numbers that appear on the blue screen:



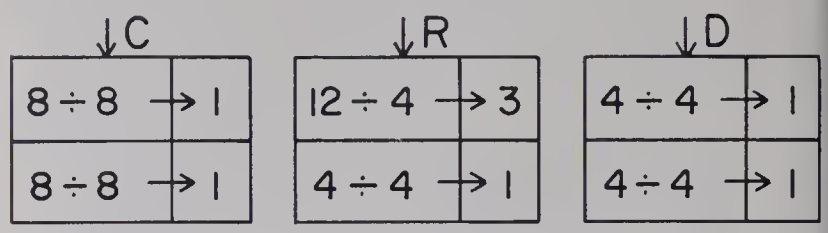


What is the pattern? It seems that the machine finds the largest common factor in the two numbers on the left, and writes the other factor on the right. The first example of R might be explained as following this pattern:



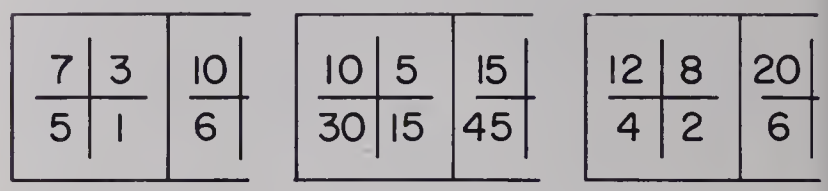
with the arrow indicating the greatest common factor of 10 and 6.

If every example is approached the same way, we can anticipate what the machine is going to do:



By what rule does the machine reach its decision to write those two numbers on the left?

Consider the first three examples under R:

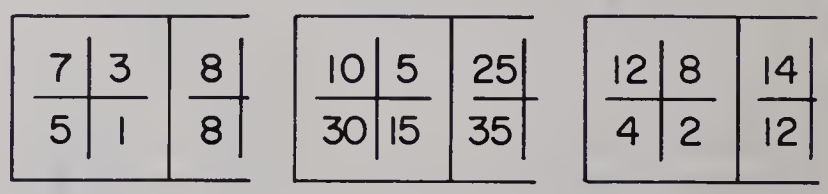


Now, rule R is obvious:

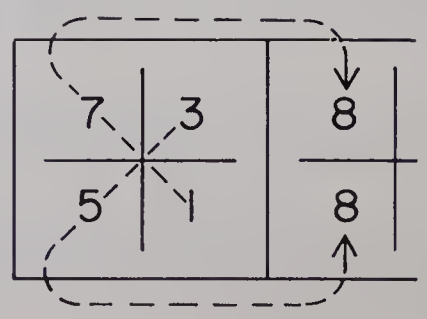
7,3 → 10
5,1 → 6
10,5 → 15
30,15 → 45
12,8 → 20
4,2 → 6
6,3 → ____
18,9 → ____

and the last two situations are found in the fourth example under R.

Consider the first three examples of Rule C in the same way:

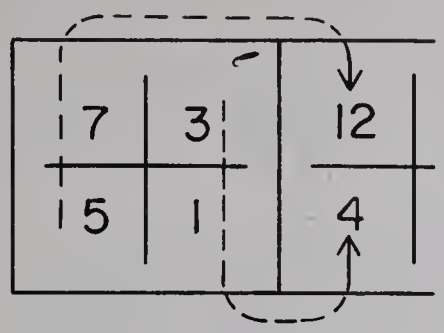


What is the pattern? We expect to see a pattern relating the pair of numbers written in the top row and the number written on the right—as in Rule R. But, none appears. Hint? Consider diagonals instead of rows.



So, Rule C can be anticipated.

Rule B doesn't seem to be concerned with rows or diagonals. Hint? Consider columns.



So, Rule B can be anticipated.

What of Rule C? Hint? Consider columns or rows or diagonals as in other rules. Look at the rows in this way:

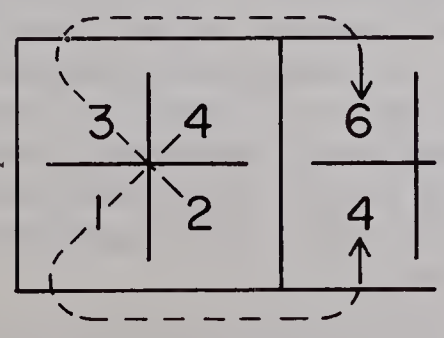
7,3 → 4
5,1 → 4
10,5 → 5
30,15 → 15
12,8 → *4*
4,2 → *2*

Aha! Rows and subtraction!

Now, all four rules on page 56 can be anticipated. With this introduction, let's study the results of the rules labeled P, F, S, and A. Rule P is revealed almost at once. Consider the rows:

3,4 → 12
1,2 → 2
1,1 → 1
3,9 → 27
5,1 → *5*
3,2 → *6*

Rule F is not so obvious. But it has a very close relationship to Rule C, when diagonals are considered:



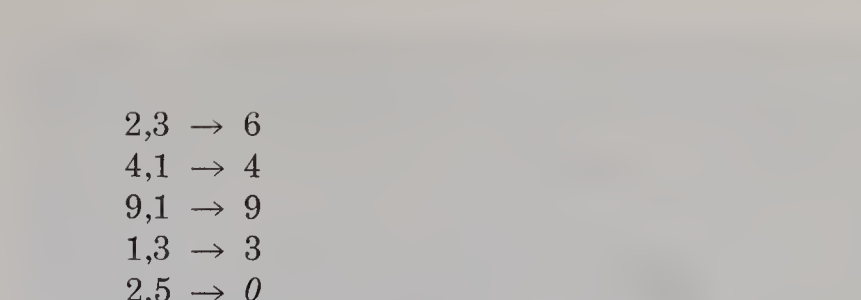
2,3 → 6
4,1 → 4
9,1 → 9
1,3 → 3
2,5 → *0*

Rules S and A are much more difficult. First, we notice a simple relationship between the pair of numbers written below the line on the left and the number written below on the right. It follows for both Rules S and A.

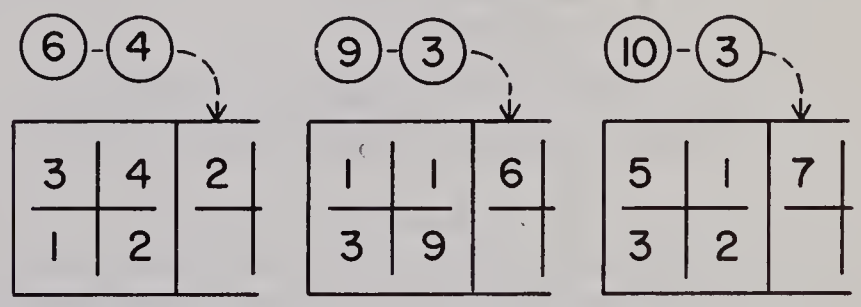
1,2 → 2
3,9 → 27
3,2 → 6
4,9 → *36*

But then the puzzle is: what pattern can we notice that will help anticipate the number written above the line on the right?

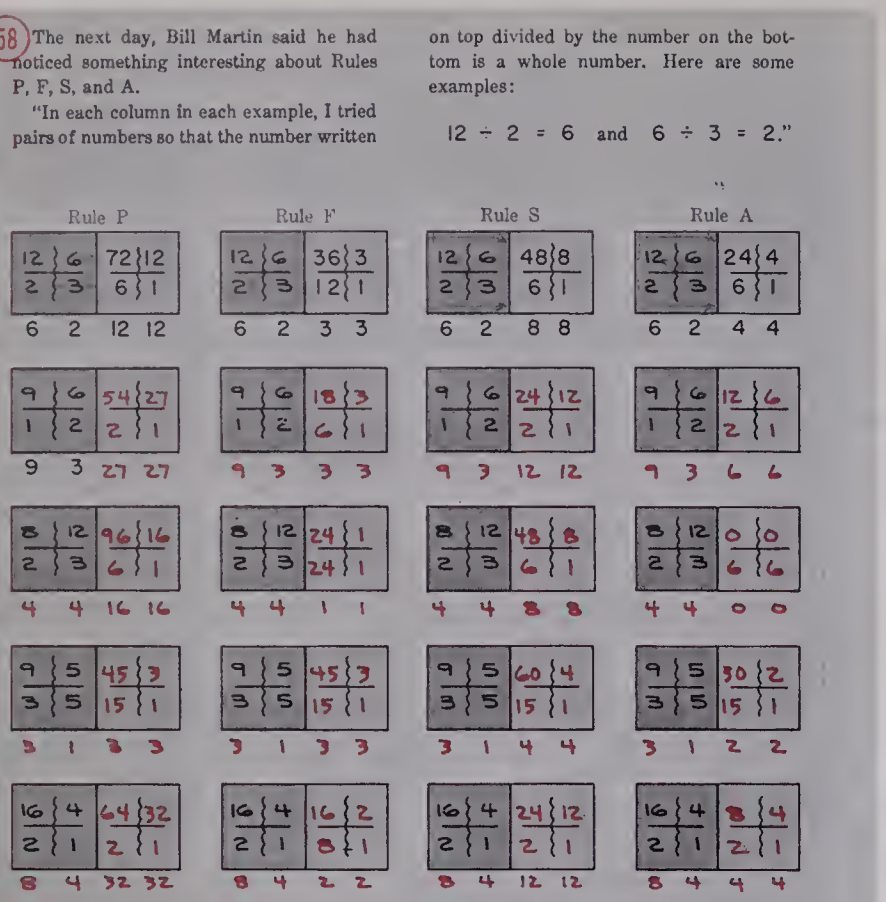
After trying many ideas—rows, columns, diagonals—we finally see a relationship when we consider both diagonals at once under Rule S:

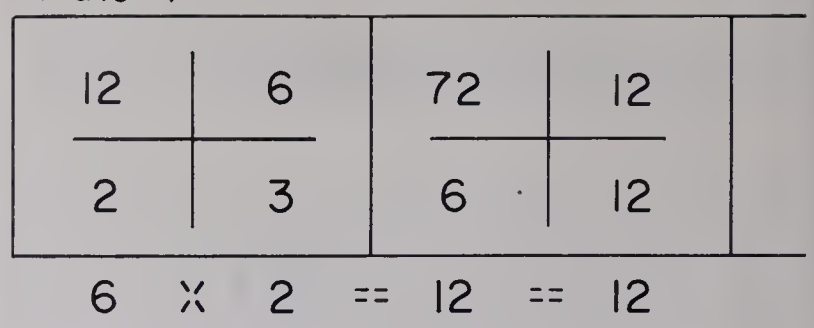


And, with this pattern in our recent experience, Rule A has a definite familiarity:



Resist the temptation to talk about these rules as anything more than eight examples of "What's my Rule?" We will, of course, return to the rules on page 57. And we hope that each child somewhere along the way will see that rules P, F, S and A are familiar if considered in a certain way. Many opportunities and clues will be provided.


58
The next day, Bill Martin said he had noticed something interesting about Rules P, F, S, and A.
"In each column in each example, I tried pairs of numbers so that the number written on top divided by the number on the bottom is a whole number. Here are some examples:
12 ÷ 2 = 6 and 6 ÷ 3 = 2."
Rule P
Rule F
Rule S
Rule A
"When I arranged the results as below, I noticed an interesting fact."
6 x 2 = 12
9 x 3 = 27
4 x 4 = 16
3 x 1 = 3
8 x 4 = 32
6 ÷ 2 = 3
9 ÷ 3 = 3
4 ÷ 4 = 1
3 ÷ 1 = 3
8 ÷ 4 = 2
6 + 2 = 8
9 + 3 = 12
4 + 4 = 8
3 + 1 = 4
8 + 4 = 12
6 - 2 = 4
9 - 3 = 6
4 - 4 = 0
3 - 1 = 2
8 - 4 = 4
58

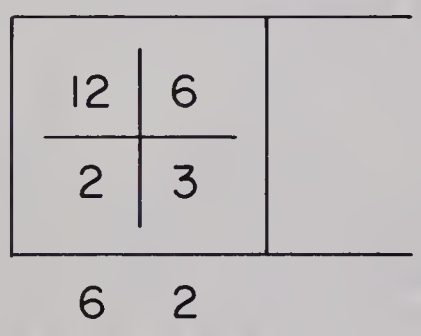


Finally, he can consider the four numbers written below as building blocks of a complete statement: $6 \times 2 = 12 = 12$—and finds a similar idea in the case of each rule. The first example, under the 4 rules, produces:

P $6 \times 2 = 12 = 12$
F $6 \div 2 = 3 = 3$
S $6 + 2 = 8 = 8$
A $6 - 2 = 4 = 4$

Notes to accompany *Work Text* page 58

Bill Martin makes an interesting discovery on page 58. He selects certain pairs of numbers to write—one member to be written above and one to be written below the line. In each case, the number written below is a factor of the one written above. He notes the other factor below the chart.



Both 2 and 6 are factors of 12; 3 and 2 are factors of 6.

Next, he applies rules P, F, S and A from page 57 to anticipate what the machine will write. His predictions are correct. Then he notices that he can extend his pattern of noting a number below the chart that, taken as a factor along with the number below the line in the chart above, leads to a product —which always is written above the line:

Jean Carlson is even more venturesome. She wonders what would happen if she considered each pair in each column as a fraction. Her results lead to some very familiar statements:

$\frac{1}{2} + \frac{1}{3} = \frac{3}{4}$
$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

These results are consistently both accurate predictions of what the machine will write and sensible and familiar patterns—at least when Rules S and A are studied.

She combined her approach with that of Bill Martin and it lead to these results:

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ $\qquad$ $\frac{1}{2} \div \frac{1}{2} = 1$
$\frac{2}{1} \times \frac{1}{3} = \frac{2}{3}$ $\qquad$ $\frac{2}{1} \div \frac{1}{3} = 6$

"Do these make sense?" "Is one-half *of* a half pie the same as a quarter of a pie?" "How many half dollars are there in a half dollar?" One. (More about this language later.) "If I walk $\frac{1}{3}$ of a mile to school and $\frac{1}{3}$ of a mile home, I walk twice the distance of $\frac{1}{3}$ mile, or $\frac{2}{3}$ of a mile." "How many $\frac{1}{3}$ hour periods (20 minutes) are there in 2 hours (120 minutes)?" Answer: 6 periods.

So, the results do seem to make some sense.

Notes to accompany *Work Text* page 59

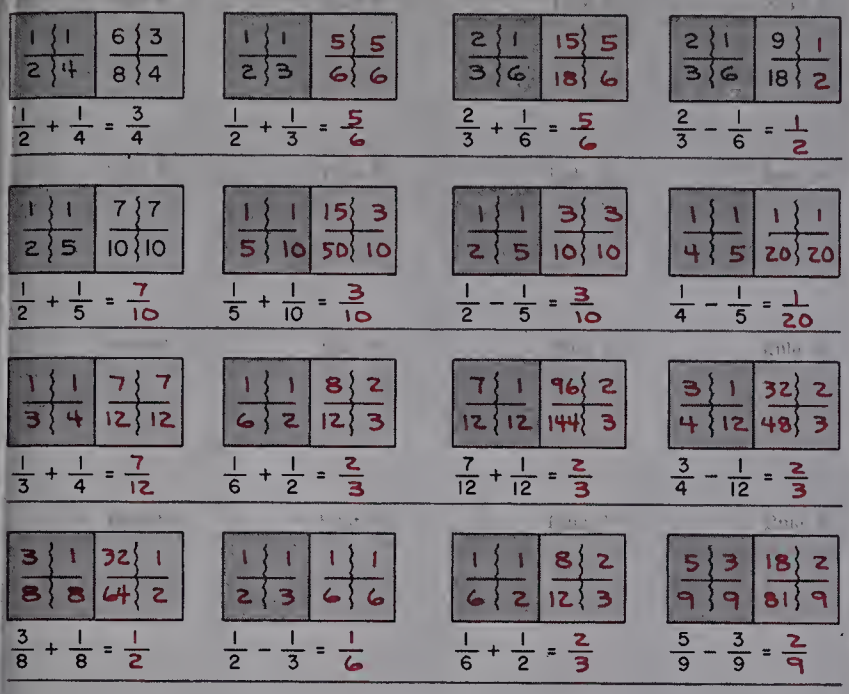
Jean Carlson then told of her own experiments.

"I went back to our earlier work in which we found some results that seemed reasonable

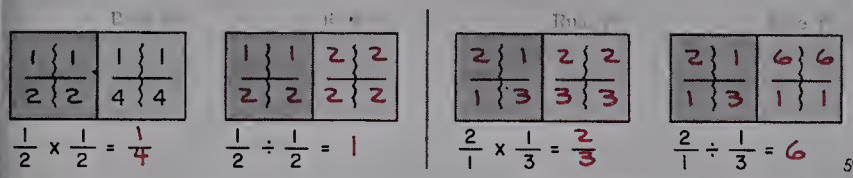
I tried Rules S and A as Alec's machine seems to operate.

"As you might expect, the machine would give exactly the same results as we got by other means.

"Try them yourselves."



"Next," Jean continued, "I tried Rule P and Rule F in the same way. The results are not so obvious."



Notes to accompany *Work Text* page 60

We have been led to suspect that Alec's machine was devised as a calculator when fractions come into the picture—multiplying (P), dividing (F), adding (S) and subtracting (A).

"Let's explore familiar expressions that involve fractions."

In each case, the pattern is suggested on the right. "Would Alec's machine give us this pattern? What rule would we use to get the same results from Alec's machine?"

$\frac{1}{2}, \frac{2}{1} \rightarrow \frac{1}{1}$ (Rule P)
$\frac{1}{2}, \frac{1}{1} \rightarrow \frac{1}{2}$ (Rule P)

In fact, each of the pattern we study on *Work Text* page 60 suggest a result we would get if we set Alec's machine to operate by Rule P and fed in the first two fractions in the expression.

So, we try to give meaning to the expressions as descriptive of real events. Try the headline story technique with some:

$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

"One-fourth of a half-inch is an eighth of an inch; and a half of a quarter inch is the same amount."

"A pound is 16 ounces. A fourth of a half-pound (8 ounces) is 2 ounces; and $\frac{1}{8}$ of a pound is 2 ounces."

$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

"One-third of a half-hour is $\frac{1}{3}$ of 30 minutes, or 10 minutes. So is $\frac{1}{6}$ of an hour 10 minutes."

"We ate a third of the half-dozen doughnuts. My sister and I had one apiece. That's 2. And 2 is $\frac{1}{6}$ of a dozen."

"It's 9 blocks to school and 9 blocks back to home —18 blocks in all. When I'm at Aunt Jane's, I'm one-third of the way to school. That's one-sixth of a round trip, and I have 15 blocks to go to complete a round trip."

"A recipe calls for $\frac{1}{2}$ cup of sugar. In one-third of a recipe, I would use $\frac{1}{6}$ of a cup of sugar."

60

Let's Talk About Dollars and Pies.	Patterns suggested by the numbers used:
Half of two dollars is ___one___ dollar.	a. $\frac{1}{2}, \frac{2}{1} \rightarrow \frac{1}{1}$
Half of one dollar is ___one-half___ of a dollar.	b. $\frac{1}{2}, \frac{1}{1} \rightarrow \frac{1}{2}$
Half of one-half dollar is ___one quarter___ of a dollar.	c. $\frac{1}{2}, \frac{1}{2} \rightarrow \frac{1}{4}$
One-fourth of eight dollars is ___two___ dollars.	d. $\frac{1}{4}, \frac{8}{1} \rightarrow \frac{2}{1}$
One-fourth of two dollars is ___one-half___ of a dollar.	e. $\frac{1}{4}, \frac{2}{1} \rightarrow \frac{1}{2}$
One-fourth of one pie is ___a fourth___ of a pie.	f. $\frac{1}{4}, \frac{1}{1} \rightarrow \frac{1}{4}$
One-fourth of one-half pie is ___an eighth___ of a pie.	g. $\frac{1}{4}, \frac{1}{2} \rightarrow \frac{1}{8}$
One-third of a half of a pie is ___a sixth___ of a pie.	h. $\frac{1}{3}, \frac{1}{2} \rightarrow \frac{1}{6}$
Let's Talk About Half-dozens.	**Patterns suggested:**
Four of them (half-dozens) is ___two___ dozen.	i. $\frac{4}{1}, \frac{1}{2} \rightarrow \frac{2}{1}$
Two of them (half-dozens) is ___one___ dozen.	j. $\frac{2}{1}, \frac{1}{2} \rightarrow \frac{1}{1}$
One-third of a half-dozen is ___one sixth___ of a dozen.	k. $\frac{1}{3}, \frac{1}{2} \rightarrow \frac{1}{6}$
Two-thirds of a half-dozen is ___one third___ (or 2/6) of a dozen.	l. $\frac{2}{3}, \frac{1}{2} \rightarrow \frac{1}{3}$

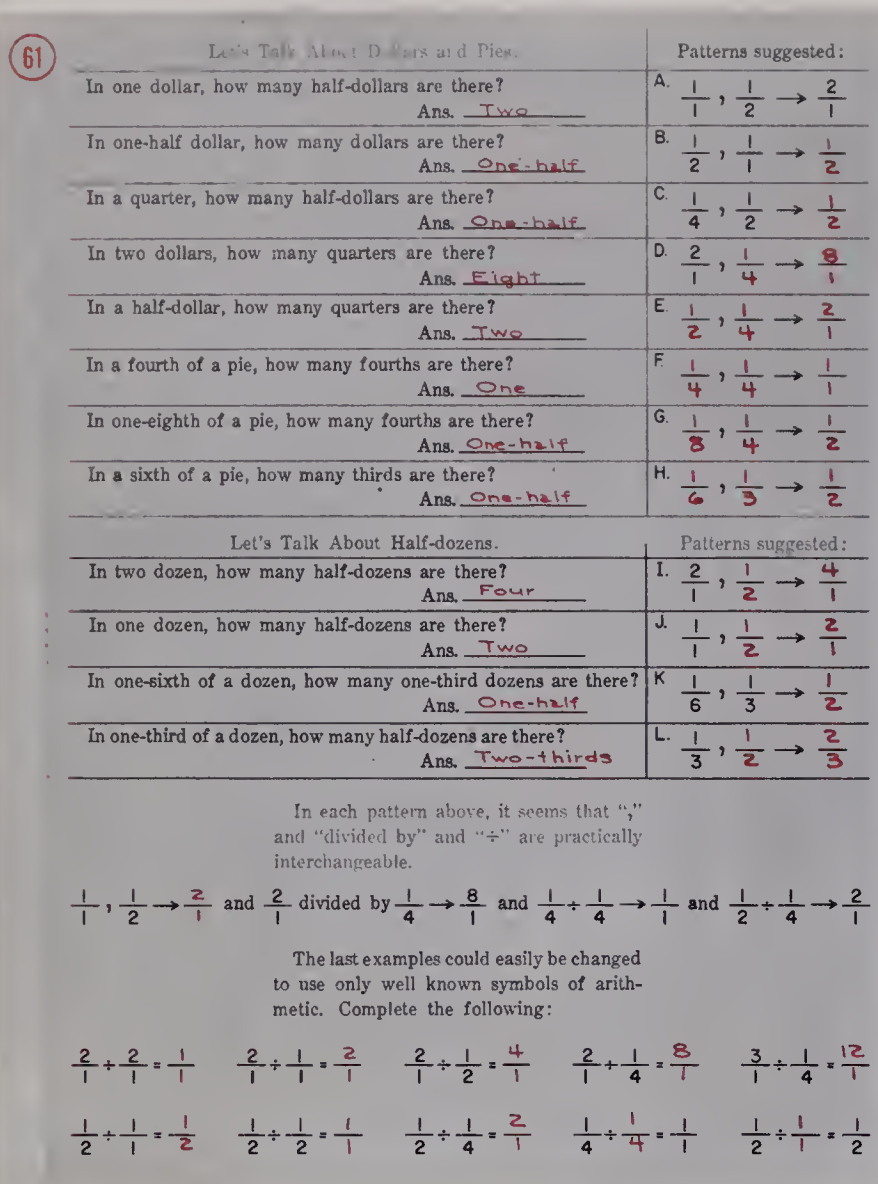
In the examples above, it seems that "," and "of" and "times" and "×" are practically interchangeable.

$\frac{1}{2}, \frac{2}{1} \rightarrow \frac{1}{1}$ and $\frac{1}{2}$ of $\frac{1}{2} \rightarrow \frac{1}{4}$ and $\frac{1}{2}$ times $\frac{1}{1} \rightarrow \frac{1}{2}$ and $\frac{1}{4} \times \frac{8}{1} \rightarrow \frac{2}{1}$

The last example could easily be changed to use only the well known symbols of arithmetic: $\frac{1}{4} \times \frac{8}{1} = \frac{2}{1}$. Complete the following:

$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$ $\frac{2}{3} \times \frac{1}{1} = \frac{2}{3}$ $\frac{1}{1} \times \frac{1}{2} = \frac{1}{2}$ $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ $\frac{2}{1} \times \frac{1}{4} = \frac{1}{2}$

$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$ $\frac{4}{1} \times \frac{1}{2} = \frac{2}{1}$ $\frac{8}{1} \times \frac{1}{4} = \frac{2}{1}$ $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$



61

Let's Talk About Dollars and Pies.

	Patterns suggested:
In one dollar, how many half-dollars are there? Ans. Two	A. $\frac{1}{1}, \frac{1}{2} \rightarrow \frac{2}{1}$
In one-half dollar, how many dollars are there? Ans. One-half	B. $\frac{1}{2}, \frac{1}{1} \rightarrow \frac{1}{2}$
In a quarter, how many half-dollars are there? Ans. One-half	C. $\frac{1}{4}, \frac{1}{2} \rightarrow \frac{1}{2}$
In two dollars, how many quarters are there? Ans. Eight	D. $\frac{2}{1}, \frac{1}{4} \rightarrow \frac{8}{1}$
In a half-dollar, how many quarters are there? Ans. Two	E. $\frac{1}{2}, \frac{1}{4} \rightarrow \frac{2}{1}$
In a fourth of a pie, how many fourths are there? Ans. One	F. $\frac{1}{4}, \frac{1}{4} \rightarrow \frac{1}{1}$
In one-eighth of a pie, how many fourths are there? Ans. One-half	G. $\frac{1}{8}, \frac{1}{4} \rightarrow \frac{1}{2}$
In a sixth of a pie, how many thirds are there? Ans. One-half	H. $\frac{1}{6}, \frac{1}{3} \rightarrow \frac{1}{2}$

Let's Talk About Half-dozens.

	Patterns suggested:
In two dozen, how many half-dozens are there? Ans. Four	I. $\frac{2}{1}, \frac{1}{2} \rightarrow \frac{4}{1}$
In one dozen, how many half-dozens are there? Ans. Two	J. $\frac{1}{1}, \frac{1}{2} \rightarrow \frac{2}{1}$
In one-sixth of a dozen, how many one-third dozens are there? Ans. One-half	K $\frac{1}{6}, \frac{1}{3} \rightarrow \frac{1}{2}$
In one-third of a dozen, how many half-dozens are there? Ans. Two-thirds	L. $\frac{1}{3}, \frac{1}{2} \rightarrow \frac{2}{3}$

In each pattern above, it seems that "," and "divided by" and "÷" are practically interchangeable.

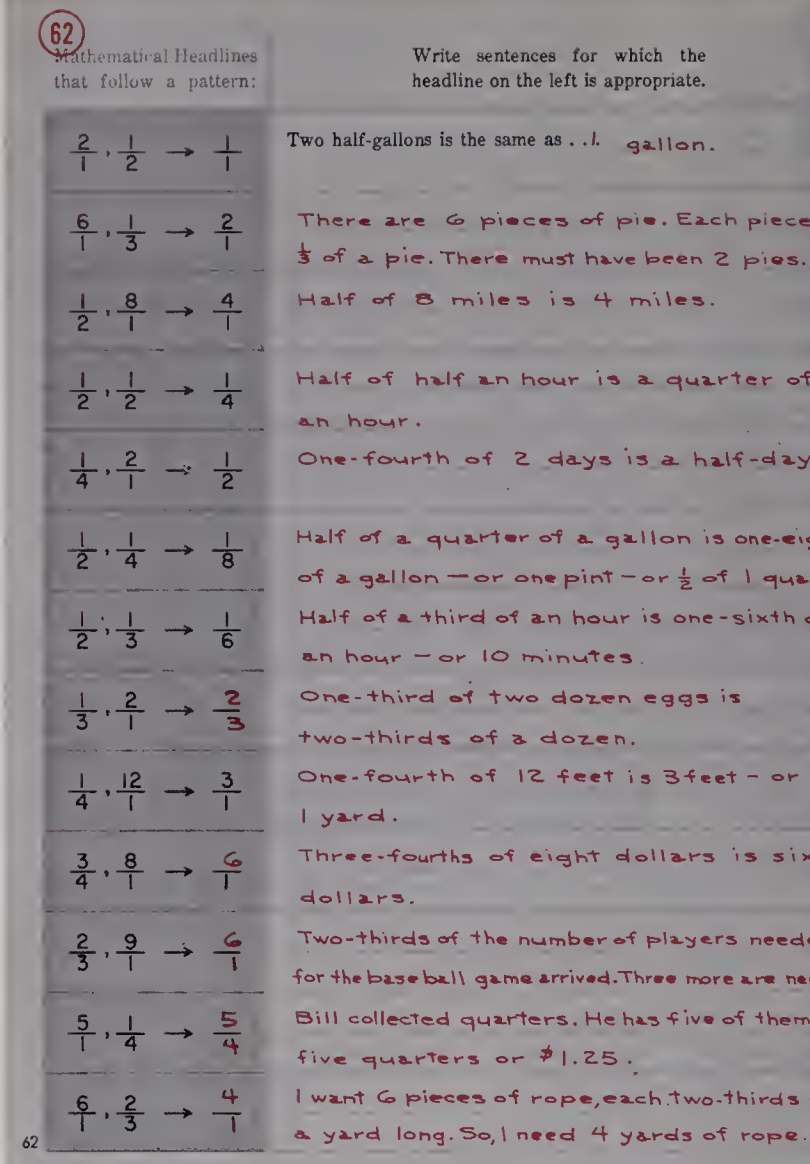
$\frac{1}{1}, \frac{1}{2} \rightarrow \frac{2}{1}$ and $\frac{2}{1}$ divided by $\frac{1}{4} \rightarrow \frac{8}{1}$ and $\frac{1}{4} \div \frac{1}{4} \rightarrow \frac{1}{1}$ and $\frac{1}{2} \div \frac{1}{4} \rightarrow \frac{2}{1}$

The last examples could easily be changed to use only well known symbols of arithmetic. Complete the following:

$\frac{2}{1} + \frac{2}{1} = \frac{1}{1}$ $\frac{2}{1} \div \frac{1}{1} = \frac{2}{1}$ $\frac{2}{1} \div \frac{1}{2} = \frac{4}{1}$ $\frac{2}{1} + \frac{1}{4} = \frac{8}{1}$ $\frac{3}{1} \div \frac{1}{4} = \frac{12}{1}$

$\frac{1}{2} \div \frac{1}{1} = \frac{1}{2}$ $\frac{1}{2} \div \frac{1}{2} = \frac{1}{1}$ $\frac{1}{2} \div \frac{1}{4} = \frac{2}{1}$ $\frac{1}{4} \div \frac{1}{4} = \frac{1}{1}$ $\frac{1}{2} \div \frac{1}{1} = \frac{1}{2}$

61



62

Mathematical Headlines that follow a pattern:

Write sentences for which the headline on the left is appropriate.

$\frac{2}{1}, \frac{1}{2} \rightarrow \frac{1}{1}$	Two half-gallons is the same as . . 1 gallon.
$\frac{6}{1}, \frac{1}{3} \rightarrow \frac{2}{1}$	There are 6 pieces of pie. Each piece is $\frac{1}{3}$ of a pie. There must have been 2 pies.
$\frac{1}{2}, \frac{8}{1} \rightarrow \frac{4}{1}$	Half of 8 miles is 4 miles.
$\frac{1}{2}, \frac{1}{2} \rightarrow \frac{1}{4}$	Half of half an hour is a quarter of an hour.
$\frac{1}{4}, \frac{2}{1} \rightarrow \frac{1}{2}$	One-fourth of 2 days is a half-day.
$\frac{1}{2}, \frac{1}{4} \rightarrow \frac{1}{8}$	Half of a quarter of a gallon is one-eighth of a gallon — or one pint — or $\frac{1}{2}$ of 1 quart.
$\frac{1}{2}, \frac{1}{3} \rightarrow \frac{1}{6}$	Half of a third of an hour is one-sixth of an hour — or 10 minutes.
$\frac{1}{3}, \frac{2}{1} \rightarrow \frac{2}{3}$	One-third of two dozen eggs is two-thirds of a dozen.
$\frac{1}{4}, \frac{12}{1} \rightarrow \frac{3}{1}$	One-fourth of 12 feet is 3 feet - or 1 yard.
$\frac{3}{4}, \frac{8}{1} \rightarrow \frac{6}{1}$	Three-fourths of eight dollars is six dollars.
$\frac{2}{3}, \frac{9}{1} \rightarrow \frac{6}{1}$	Two-thirds of the number of players needed for the baseball game arrived. Three more are needed.
$\frac{5}{1}, \frac{1}{4} \rightarrow \frac{5}{4}$	Bill collected quarters. He has five of them — five quarters or $1.25.
$\frac{6}{1}, \frac{2}{3} \rightarrow \frac{4}{1}$	I want 6 pieces of rope, each two-thirds of a yard long. So, I need 4 yards of rope.

62

Notes to accompany *Work Text* pages 61-63

A second list of translations that is somewhat new, when we explore patterns of life and of arithmetic involving fractions, is illustrated by $2 \div \frac{1}{2}$: "two divided by a half . . ." doesn't ring true at first. And then we meet $\frac{1}{4} \div \frac{1}{2}$: "one-fourth divided by one-half." To translate these expressions so they are clear, we need to change the language a little: $2 \div \frac{1}{2}$ and $\frac{1}{4} \div \frac{1}{2}$ become "How many halves are there in two?" . . . and "How many halves are there in a quarter?"

More concretely: "How many half-dollars are there in 2 dollars?" . . . and "How many half-dollars are there in 25¢?"

Now, the answers $2 \div \frac{1}{2} = 4$ and $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$ are reasonable.

This problem occurs less often when we are using whole numbers. "If a group of 12 is divided into groups of 3, how many groups will there be?" $12 \div 4 = 3$; the translation is direct and immediately reasonable.

But, "How much is 12 inches divided by $\frac{1}{2}$ inch" needs some rearrangement: "How many half-inches are there in a foot?" Now, $12 \div \frac{1}{2} = 24$ is clearly a reasonable way to reflect the idea.

Work Text page 61 assumes we can take this liberty in moving back and forth between the real world and expressions of arithmetic. And, as we develop certain patterns such as "$\frac{1}{6}, \frac{1}{3} \rightarrow \frac{1}{2}$" we find that Alec's machine set at Rule F would produce similar results.

Finally, we write $\frac{2}{1} \div \frac{1}{4} = \frac{9}{1}$—and move directly to try it as a headline:

$\frac{2}{1} \div \frac{1}{4} = \frac{8}{1}$

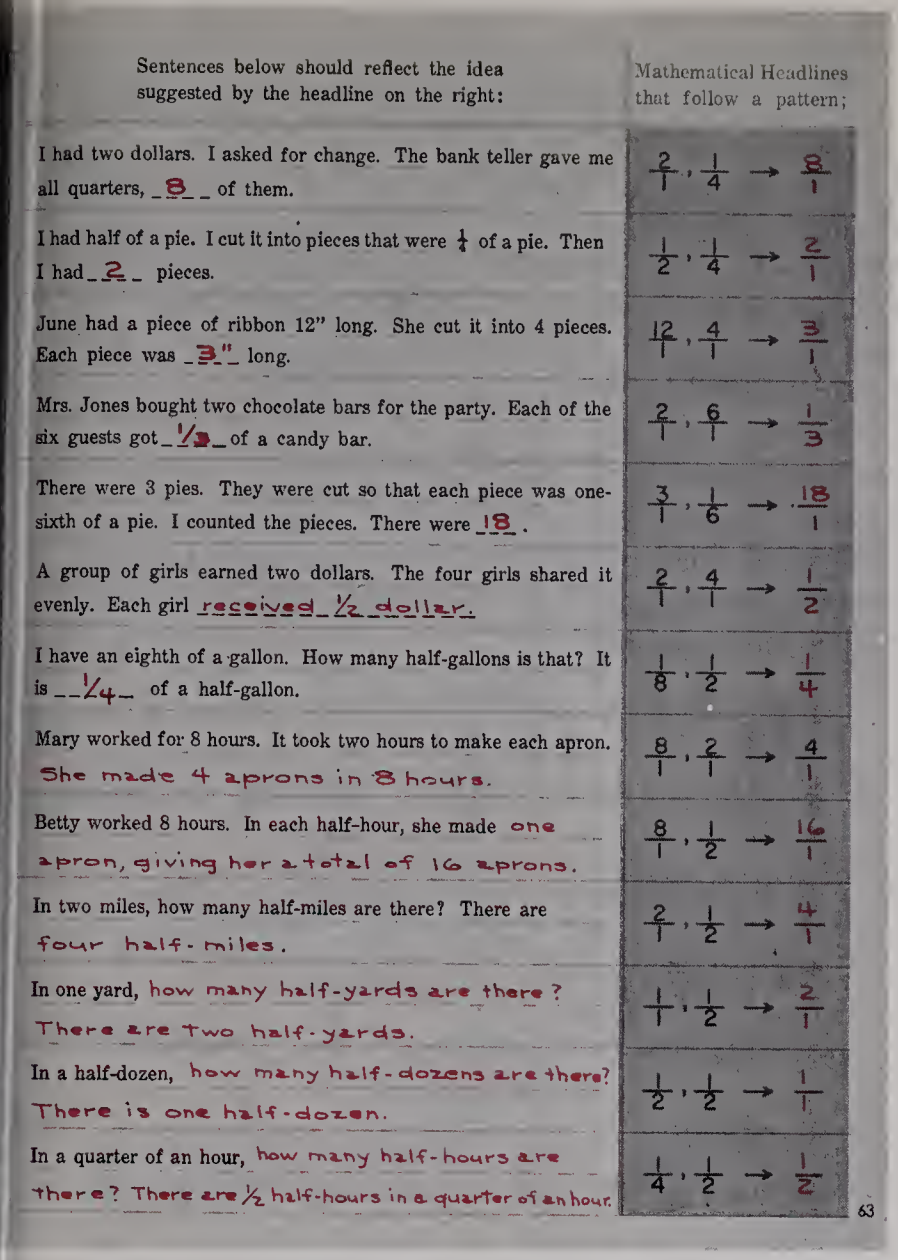
"In two dollars, there are 8 quarters."

"If 2 feet of ribbon are cut into 3 inch, or $\frac{1}{4}$ foot, lengths, there will be 8 pieces."

"There are 8 periods of 15 minutes each in 2 hours."

"If I pour 2 gallons of milk into quart bottles, I can fill 8 quart bottles."

After several such examples, the basis is established for two pages of independent work—writing



Sentences below should reflect the idea suggested by the headline on the right:

Mathematical Headlines that follow a pattern;

I had two dollars. I asked for change. The bank teller gave me all quarters, _8_ of them. — $\frac{2}{1}, \frac{1}{4} \rightarrow \frac{8}{1}$

I had half of a pie. I cut it into pieces that were $\frac{1}{4}$ of a pie. Then I had _2_ pieces. — $\frac{1}{2}, \frac{1}{4} \rightarrow \frac{2}{1}$

June had a piece of ribbon 12" long. She cut it into 4 pieces. Each piece was _3"_ long. — $\frac{12}{1}, \frac{4}{1} \rightarrow \frac{3}{1}$

Mrs. Jones bought two chocolate bars for the party. Each of the six guests got _1/3_ of a candy bar. — $\frac{2}{1}, \frac{6}{1} \rightarrow \frac{1}{3}$

There were 3 pies. They were cut so that each piece was one-sixth of a pie. I counted the pieces. There were _18_. — $\frac{3}{1}, \frac{1}{6} \rightarrow \frac{18}{1}$

A group of girls earned two dollars. The four girls shared it evenly. Each girl _received ½ dollar._ — $\frac{2}{1}, \frac{4}{1} \rightarrow \frac{1}{2}$

I have an eighth of a gallon. How many half-gallons is that? It is _1/4_ of a half-gallon. — $\frac{1}{8}, \frac{1}{2} \rightarrow \frac{1}{4}$

Mary worked for 8 hours. It took two hours to make each apron. She made 4 aprons in 8 hours. — $\frac{8}{1}, \frac{2}{1} \rightarrow \frac{4}{1}$

Betty worked 8 hours. In each half-hour, she made one apron, giving her a total of 16 aprons. — $\frac{8}{1}, \frac{1}{2} \rightarrow \frac{16}{1}$

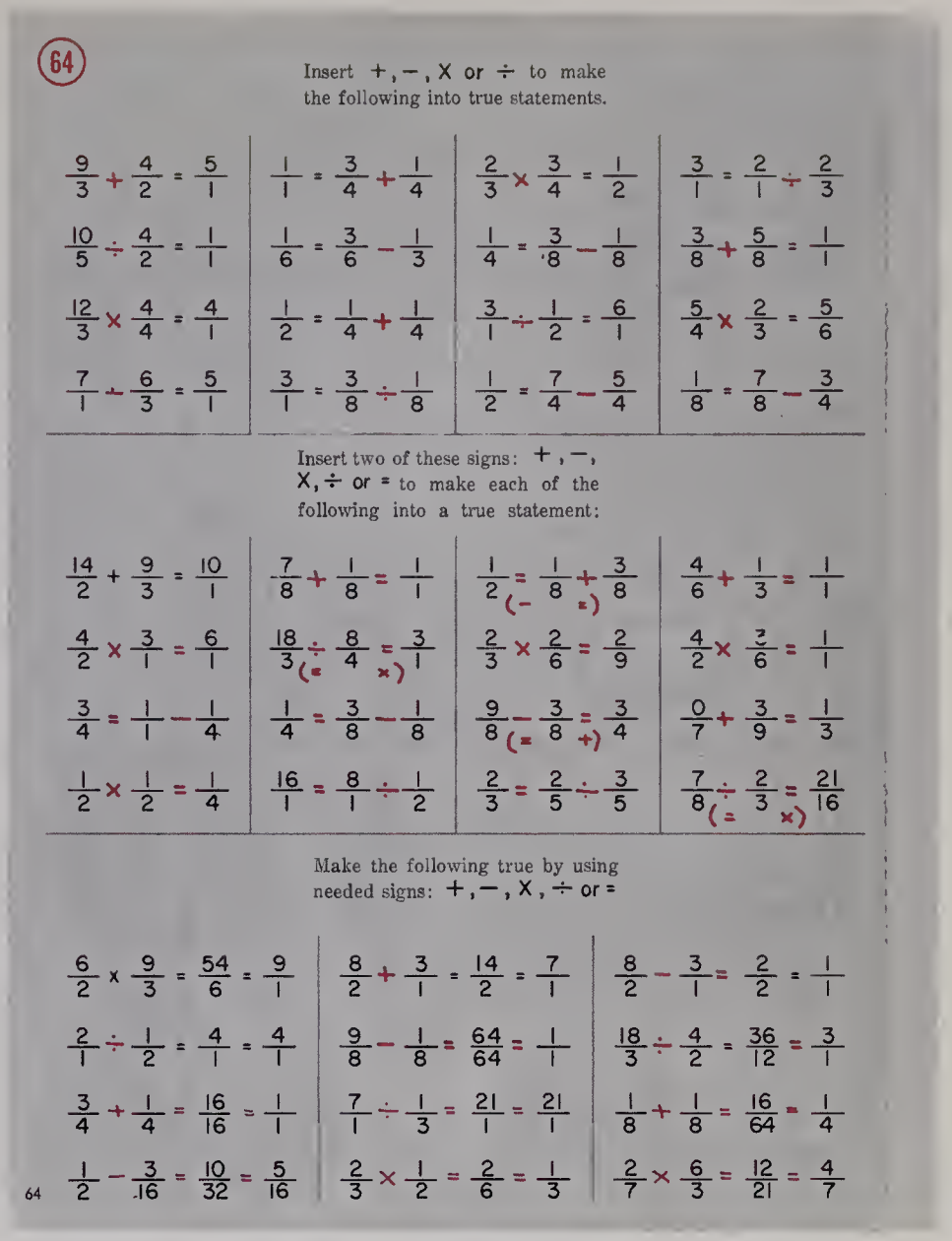
In two miles, how many half-miles are there? There are four half-miles. — $\frac{2}{1}, \frac{1}{2} \rightarrow \frac{4}{1}$

In one yard, how many half-yards are there? There are two half-yards. — $\frac{1}{1}, \frac{1}{2} \rightarrow \frac{2}{1}$

In a half-dozen, how many half-dozens are there? There is one half-dozen. — $\frac{1}{2}, \frac{1}{2} \rightarrow \frac{1}{1}$

In a quarter of an hour, how many half-hours are there? There are ½ half-hours in a quarter of an hour. — $\frac{1}{4}, \frac{1}{2} \rightarrow \frac{1}{2}$

63



64

Insert +, −, X or ÷ to make the following into true statements.

$\frac{9}{3} + \frac{4}{2} = \frac{5}{1}$	$\frac{1}{1} = \frac{3}{4} + \frac{1}{4}$	$\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$	$\frac{3}{1} = \frac{2}{1} \div \frac{2}{3}$
$\frac{10}{5} \div \frac{4}{2} = \frac{1}{1}$	$\frac{1}{6} = \frac{3}{6} - \frac{1}{3}$	$\frac{1}{4} = \frac{3}{8} - \frac{1}{8}$	$\frac{3}{8} + \frac{5}{8} = \frac{1}{1}$
$\frac{12}{3} \times \frac{4}{4} = \frac{4}{1}$	$\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$	$\frac{3}{1} \div \frac{1}{2} = \frac{6}{1}$	$\frac{5}{4} \times \frac{2}{3} = \frac{5}{6}$
$\frac{7}{1} - \frac{6}{3} = \frac{5}{1}$	$\frac{3}{1} = \frac{3}{8} \div \frac{1}{8}$	$\frac{1}{2} = \frac{7}{4} - \frac{5}{4}$	$\frac{1}{8} = \frac{7}{8} - \frac{3}{4}$

Insert two of these signs: +, −, X, ÷ or = to make each of the following into a true statement:

$\frac{14}{2} + \frac{9}{3} = \frac{10}{1}$	$\frac{7}{8} + \frac{1}{8} = \frac{1}{1}$	$\frac{1}{2} = \frac{1}{8} + \frac{3}{8}$ (−, =)	$\frac{4}{6} + \frac{1}{3} = \frac{1}{1}$
$\frac{4}{2} \times \frac{3}{1} = \frac{6}{1}$	$\frac{18}{3} \div \frac{8}{4} = \frac{3}{1}$ (=, ×)	$\frac{2}{3} \times \frac{2}{6} = \frac{2}{9}$	$\frac{4}{2} \times \frac{3}{6} = \frac{1}{1}$
$\frac{3}{4} = \frac{1}{1} - \frac{1}{4}$	$\frac{1}{4} = \frac{3}{8} - \frac{1}{8}$	$\frac{9}{8} - \frac{3}{8} = \frac{3}{4}$ (=, +)	$\frac{0}{7} + \frac{3}{9} = \frac{1}{3}$
$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	$\frac{16}{1} = \frac{8}{1} \div \frac{1}{2}$	$\frac{2}{3} = \frac{2}{5} \div \frac{3}{5}$	$\frac{7}{8} \div \frac{2}{3} = \frac{21}{16}$ (=, ×)

Make the following true by using needed signs: +, −, X, ÷ or =

$\frac{6}{2} \times \frac{9}{3} = \frac{54}{6} = \frac{9}{1}$	$\frac{8}{2} + \frac{3}{1} = \frac{14}{2} = \frac{7}{1}$	$\frac{8}{2} - \frac{3}{1} = \frac{2}{2} = \frac{1}{1}$
$\frac{2}{1} \div \frac{1}{2} = \frac{4}{1} = \frac{4}{1}$	$\frac{9}{8} - \frac{1}{8} = \frac{64}{64} = \frac{1}{1}$	$\frac{18}{3} \div \frac{4}{2} = \frac{36}{12} = \frac{3}{1}$
$\frac{3}{4} + \frac{1}{4} = \frac{16}{16} = \frac{1}{1}$	$\frac{7}{1} \div \frac{1}{3} = \frac{21}{1} = \frac{21}{1}$	$\frac{1}{8} + \frac{1}{8} = \frac{16}{64} = \frac{1}{4}$
$\frac{1}{2} - \frac{3}{16} = \frac{10}{32} = \frac{5}{16}$	$\frac{2}{3} \times \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$	$\frac{2}{7} \times \frac{6}{3} = \frac{12}{21} = \frac{4}{7}$

64

stories to match headlines, and completing headlines to fit stories on *Work Text* pages 62 and 63.

Notes to accompany *Work Text* pages 64-67

This series of four pages—64, 65, 66, and 67—is designed to lay a strong intuitive understanding of division in which the given factor is a fraction and in which both the product and given factor are fractions.

As these four pages are nearing completion, we hope that children are prepared to bump into $\frac{7}{8} \div \frac{2}{3} = ?$ on page 67 before any attention has been paid to *manipulating* fractions. We hope to keep the intuitive approach alive as long as we can, with children searching each step of the way for real-life patterns that are expressed by such mathematical statements as $\frac{2}{3} \div \frac{1}{6} = 4$: "Two-thirds of a yard is 2 feet or 24 inches; $\frac{1}{6}$ of a yard is 6 inches, or a half foot; since $24 \div 6 = 4$, I could cut a board $\frac{2}{3}$ of a yard long into 4 pieces, each of which would be $\frac{1}{6}$ of a yard long."

There is no more firm foundation to support the trips we must take into the abstract when we argue in any of the following ways:

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \times \frac{3}{2} = \frac{21}{16} = 1\frac{5}{16}$$

("Invert and multiply")

$$\frac{7}{8} \div \frac{2}{3} = \frac{21}{24} \div \frac{16}{21} = \frac{21}{16} = 1\frac{5}{16}$$

("least common denominator")

$$\frac{7}{8} \div \frac{2}{3} = \frac{7}{8} \times \frac{3}{2} \div \frac{2}{3} \times \frac{3}{2} = \frac{7}{8} \times \frac{3}{2} \div 1 = \frac{7}{8} \times \frac{3}{2} = \frac{21}{16} = 1\frac{5}{16}$$

("multiplication inverse")

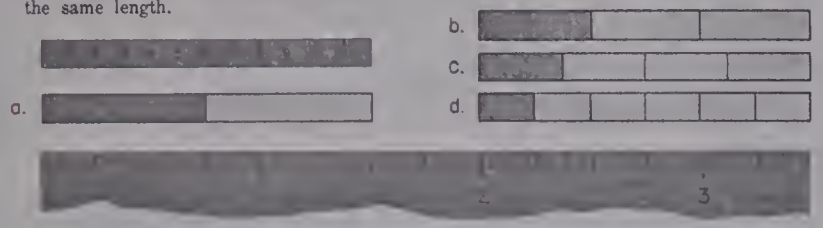
$$\frac{7}{8} \div \frac{2}{3} = \frac{\frac{7}{8}}{\frac{2}{3}} = \frac{\frac{7}{8}}{\frac{2}{3}} \times \frac{\frac{3}{2}}{\frac{3}{2}} = \frac{\frac{7}{8} \times \frac{3}{2}}{\frac{2}{3} \times \frac{3}{2}} = \frac{\frac{7}{8} \times \frac{3}{2}}{1} = \frac{7}{8} \times \frac{3}{2} = \frac{21}{16} = 1\frac{5}{16}$$

(Another form of "multiplication inverse")

65

Consider some of the many ways in which you could cut up a ruler into pieces that are the same length.

Different possibilities can be summarized in the chart below:



Ways of cutting a 12-inch ruler into pieces of the same length.

Length of pieces	6 inches	4 inches	12 inches	3 inches	2 inches	1 inch	$\frac{1}{2}$ inch	$\frac{1}{4}$ inch	$\frac{1}{8}$ inch	$\frac{1}{3}$ inch	$\frac{2}{3}$ inch	$\frac{3}{4}$ inch
Number of pieces	2	3	1	4	6	12	24	48	96	36	18	16

Complete the following:

In twelve dollars there are 24 half-dollars and 48 quarters and 120 dimes and 240 nickels and 1200 pennies.

If you can make a jump on the moon $\frac{1}{4}$ of a mile long, it would take 4 jumps to go 1 mile, 2 jumps to go a half-mile, 1 jump to go a quarter of a mile, 8 jumps to go two miles, 16 jumps to go 4 miles, and 12 jumps to go 3 miles.

If 12 gallons of milk is poured into half-gallon containers, it would fill 24 of them, or it would fill 12 one-gallon containers, or 48 quart containers, or 96 pint containers.

If a yardstick is cut up into pieces of equal length, there would be 12 pieces 3 inches long, 72 pieces $\frac{1}{2}$ inch long, or 18 pieces 2 in. long, and 144 pieces $\frac{1}{4}$ inch long.

Write examples to fit the following headlines:

$24 \div \frac{1}{2} = 48$	In 2 dozen pairs of shoes there are 48 shoes.
At a bank, I could change $24.00 into 48 half-dollars.	
$6 = 2 \div \frac{1}{3}$	I cut 6 pieces, the same weight, out of a 2 pound piece of cheese. Each piece weighed $\frac{1}{3}$ of a pound.
$4 \div \frac{2}{3} = 6$	On a 4 mile hike, I rested every $\frac{2}{3}$ of a mile. There are 6 two-thirds of a mile in 4, so I must have rested 5 times.

65

No matter which of these approaches we develop, we must realize that each requires a bit of faith on the part of children—each involves a step or two we would not like to discuss (at this time) with full mathematical rigor.

The best support for any of these algorisms will be that they always produce reasonable results—as long as we have an intuitive idea of what the result ought to be. All the procedures used above are extended trips through the abstract and, at times, defy reference to the concrete aspects of the problem.

When we begin trying to draw a sketch to illustrate the idea of $\frac{7}{8} \div \frac{2}{3}$, it becomes very complicated. Then we try to illustrate $\frac{2}{3} \div \frac{7}{8}$, and further complications set in—and $\frac{3}{7} \div \frac{4}{9}$ multiplies the complexities of the sketch. It takes some preliminary study to decide whether the answer is going to be greater or less than 1.

Consider the following approach: Divide the class into five committees to study five different proposals that have been suggested by Mary Jones, Bill Dunn, Ruth Knowles, Fred Stokes and Alec Marson. The story is that five children made up different sets of rules as shortcuts when they encountered a problem involving division of fractions.

"Let's study each of these shortcuts to see if they are reliable. In the last eight *Work Text* pages, we have found many reasonable answers. Suppose we try each shortcut in each of those examples. If the shortcuts lead to contradictory conclusions, we will certainly not depend on it in the future. Then we can devise other tests for each shortcut. The five proposals will be referred to by the author's name."

Mary's rule

"Invert the divisor and multiply!" Mary, of course, explained what each of her terms meant and gave examples to illustrate. In the following, the "divisor" is the given factor:

$$3 \div \frac{2}{3} = \qquad \frac{4}{7} \div \frac{1}{5} = \qquad \frac{3}{4} \div 6 =$$

By "inverting" the divisor, she gave these examples:

$$\tfrac{2}{3} \rightarrow \tfrac{3}{2}$$
$$\tfrac{4}{5} \rightarrow \tfrac{5}{1}$$
$$\tfrac{6}{1} \rightarrow \tfrac{1}{6}$$

By "multiply" she means to change the division sign to a multiplication sign. Finally, applying her rule, she argues:

$$3 \div \frac{2}{3} = \frac{3}{1} \times \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$$

$$\frac{4}{7} \div \frac{1}{5} = \frac{4}{7} \times \frac{5}{1} = \frac{20}{7} = 2\frac{6}{7}$$

$$\frac{3}{4} \div 6 = \frac{3}{4} \times \frac{1}{6} = \frac{3}{24} = \frac{1}{8}$$

"Let's have some volunteers to test 'Mary's Rule.' "

Bill's rule

"Find the least common denominator." Bill argued that in addition and subtraction, we find a least common denominator, so try the same in division. He used the same examples as Mary:

$$3 \div \frac{2}{3} = \frac{3}{1} \div \frac{2}{3} = \frac{9}{3} \div \frac{2}{3} = 9 \div 2 = 4\frac{1}{2}$$

$$\frac{4}{7} \div \frac{1}{5} = \frac{20}{35} \div \frac{7}{35} = 20 \div 7 = 2\frac{6}{7}$$

$$\frac{3}{4} \div 6 = \frac{3}{4} \div \frac{6}{1} = \frac{3}{4} \div \frac{24}{4} = 3 \div 24 = \frac{1}{8}$$

"Volunteers to test Bill's rule?"

Ruth's rule

"Multiply the product and given factor by some number so that the divisor turns out to be 1." Again, she used the same examples:

$$3 \div \frac{2}{3} = \frac{3}{1} \times \frac{3}{2} \div \boxed{\frac{2}{3} \times \frac{3}{2}} =$$

Now she explains that

$$\frac{2}{3} \times \frac{3}{2} = \frac{2 \times 3}{3 \times 2} = \frac{6}{6} = 1$$

So, she argued that

$$\frac{3}{1} \times \frac{3}{2} \div \boxed{\frac{2}{3} \times \frac{3}{2}} = \frac{3}{1} \times \frac{3}{2} \div 1 =$$

At this point, she claimed that any number divided by one leaves the number unchanged, so:

$$\frac{3}{1} \times \frac{3}{2} \div 1 = \frac{3}{1} \times \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$$

She used a little shorthand to simplify her work:

$$\frac{4}{7} \div \frac{1}{5} = \frac{4}{7} \times \frac{5}{1} \div \boxed{\frac{1}{5} \times \frac{5}{1}} = \frac{4}{7} \times \frac{5}{1} = \frac{20}{7} = 2\frac{6}{7}$$

$$\frac{3}{4} \div 6 = \frac{3}{4} \times \frac{1}{6} \div \boxed{6 \times \frac{1}{6}} = \frac{3}{4} \times \frac{1}{6} = \frac{3}{24} = \frac{1}{8}$$

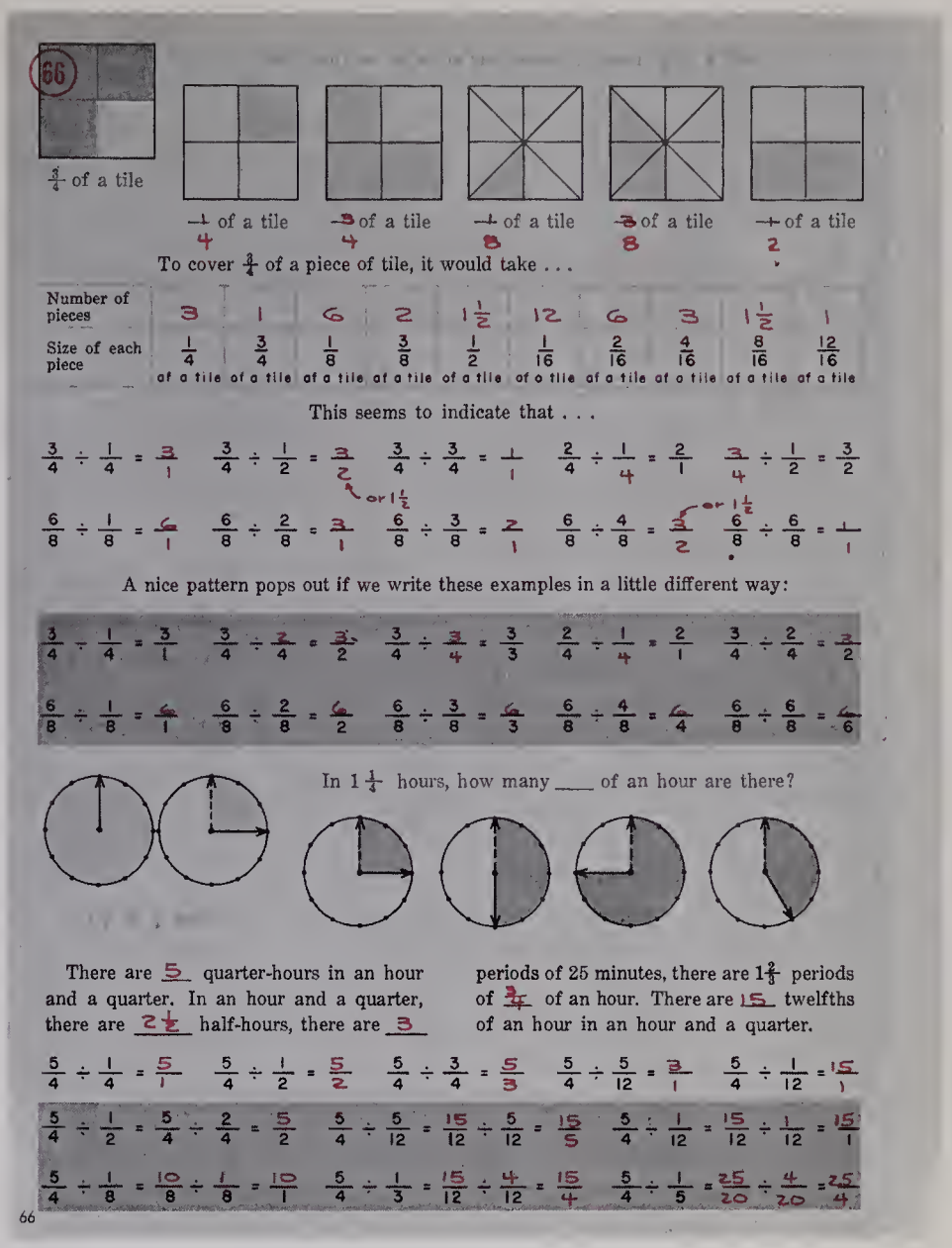
So, Ruth gets the same results that Mary and Bill found.

"Have we volunteers to test Ruth's rule?"

Fred's rule

"Write the division as a complex fraction, and then multiply by a particular name for 1."

$$3 \div \frac{2}{3} = \frac{\frac{3}{1}}{\frac{2}{3}} \times \boxed{\frac{\frac{3}{2}}{\frac{3}{2}}} =$$


66
¾ of a tile
¼ of a tile
¾ of a tile
⅛ of a tile
¾ of a tile
½ of a tile
To cover ¾ of a piece of tile, it would take . . .
Number of pieces
Size of each piece
This seems to indicate that . . .
A nice pattern pops out if we write these examples in a little different way:
In 1¼ hours, how many ___ of an hour are there?
There are 5 quarter-hours in an hour and a quarter. In an hour and a quarter, there are 2½ half-hours, there are 3
periods of 25 minutes, there are 1⅔ periods of ¾ of an hour. There are 15 twelfths of an hour in an hour and a quarter.

Fred argues that he could do this because:

a. any number multiplied by 1 leaves the number unchanged, and

b. $\frac{3}{2}$ or $\frac{3}{2} \div \frac{3}{2} = 1$, because any number divided by the same number is another name for 1.

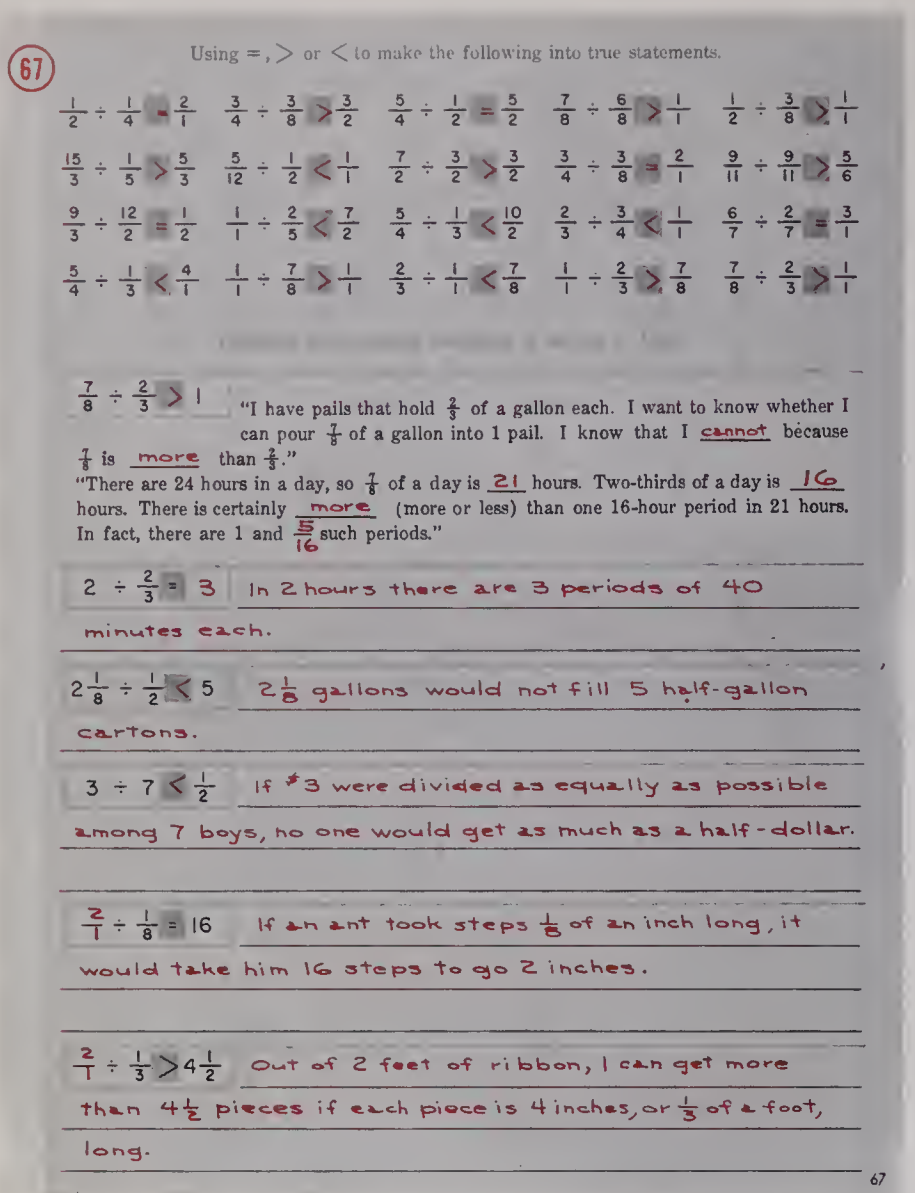
So Fred went on:

$$\frac{\frac{3}{1}}{\frac{2}{3}} \times \frac{\frac{3}{2}}{\frac{3}{2}} = \frac{\frac{3}{1} \times \frac{3}{2}}{\boxed{\frac{2}{3} \times \frac{3}{2}}}$$

But $\frac{2}{3} \times \frac{3}{2} = \frac{2 \times 3}{3 \times 2} = \frac{6}{6} = 1$, so

$$\frac{\frac{3}{1} \times \frac{3}{2}}{\boxed{\frac{2}{3} \times \frac{3}{2}}} = \frac{\frac{3}{1} \times \frac{3}{2}}{1}$$

Finally, to divide any number by 1 leaves it unchanged, so:



(67)

Using =, > or < to make the following into true statements.

$\frac{1}{2} \div \frac{1}{4} = \frac{2}{1}$ $\frac{3}{4} \div \frac{3}{8} > \frac{3}{2}$ $\frac{5}{4} \div \frac{1}{2} = \frac{5}{2}$ $\frac{7}{8} \div \frac{6}{8} > \frac{1}{1}$ $\frac{1}{2} \div \frac{3}{8} > \frac{1}{1}$

$\frac{15}{3} \div \frac{1}{5} > \frac{5}{3}$ $\frac{5}{12} \div \frac{1}{2} < \frac{1}{1}$ $\frac{7}{2} \div \frac{3}{2} > \frac{3}{2}$ $\frac{3}{4} \div \frac{3}{8} = \frac{2}{1}$ $\frac{9}{11} \div \frac{9}{11} > \frac{5}{6}$

$\frac{9}{3} \div \frac{12}{2} = \frac{1}{2}$ $\frac{1}{1} \div \frac{2}{5} < \frac{7}{2}$ $\frac{5}{4} \div \frac{1}{3} < \frac{10}{2}$ $\frac{2}{3} \div \frac{3}{4} < \frac{1}{1}$ $\frac{6}{7} \div \frac{2}{7} = \frac{3}{1}$

$\frac{5}{4} \div \frac{1}{3} < \frac{4}{1}$ $\frac{1}{1} \div \frac{7}{8} > \frac{1}{1}$ $\frac{2}{3} \div \frac{1}{1} < \frac{7}{8}$ $\frac{1}{1} \div \frac{2}{3} > \frac{7}{8}$ $\frac{7}{8} \div \frac{2}{3} > \frac{1}{1}$

$\frac{7}{8} \div \frac{2}{3} > 1$ "I have pails that hold $\frac{2}{3}$ of a gallon each. I want to know whether I can pour $\frac{7}{8}$ of a gallon into 1 pail. I know that I cannot because $\frac{7}{8}$ is more than $\frac{2}{3}$."

"There are 24 hours in a day, so $\frac{7}{8}$ of a day is 21 hours. Two-thirds of a day is 16 hours. There is certainly more (more or less) than one 16-hour period in 21 hours. In fact, there are 1 and $\frac{5}{16}$ such periods."

$2 \div \frac{2}{3} = 3$ In 2 hours there are 3 periods of 40 minutes each.

$2\frac{1}{8} \div \frac{1}{2} < 5$ $2\frac{1}{8}$ gallons would not fill 5 half-gallon cartons.

$3 \div 7 < \frac{1}{2}$ If $3 were divided as equally as possible among 7 boys, no one would get as much as a half-dollar.

$\frac{2}{1} \div \frac{1}{8} = 16$ If an ant took steps $\frac{1}{8}$ of an inch long, it would take him 16 steps to go 2 inches.

$\frac{2}{1} \div \frac{1}{3} > 4\frac{1}{2}$ Out of 2 feet of ribbon, I can get more than $4\frac{1}{2}$ pieces if each piece is 4 inches, or $\frac{1}{3}$ of a foot, long.

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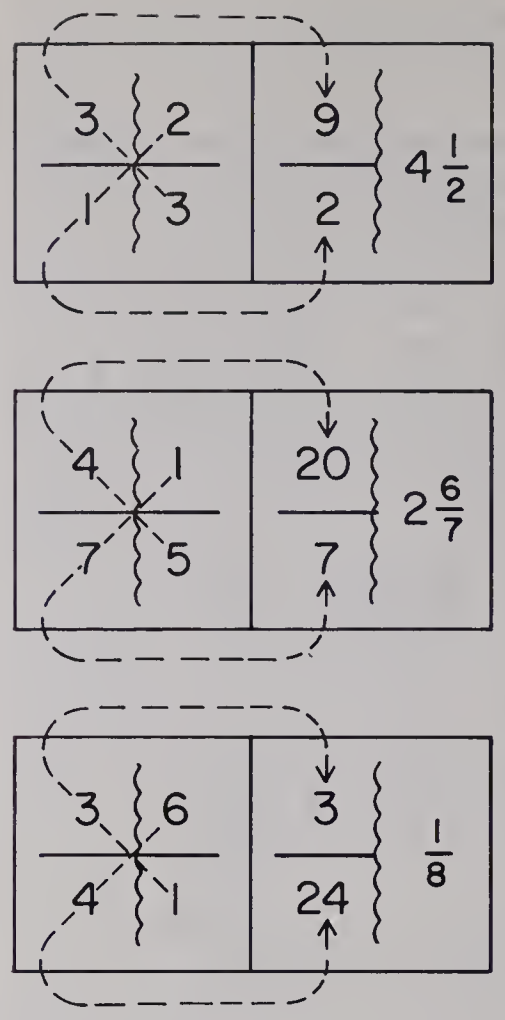


3 x 3 = 9
2 x 1 = 2

5 x 4 = 20
1 x 7 = 7

1 x 3 = 3
6 x 4 = 24

"Are there any volunteers to test Alec's rule F?"

Other tests

Each group tests one of the five rules to find if it produces results that have been found on recent pages. Then suggest another test: Does the rule make sense when we consider whole numbers? How about such examples as the following:

$100 \div 4 = 25$	$1{,}000 \div 10 = 100$
$9 \div 3 = 3$	$18 \div 5 = 3\frac{3}{5}$
$1 \div 2 = \frac{1}{2}$	$37 \div 1 = 37$

"Here is another test. In division, we are finding a missing factor. If our results are reliable, then we can multiply the given and "found" factor to find the product which we were also given.

"If a rule argues that $\frac{7}{8} \div \frac{2}{3} = 1\frac{5}{16}$, then we can test by finding whether it is true that $1\frac{5}{16} \times \frac{2}{3} = \frac{7}{8}$."

"Make up some examples of divisions that involve fractions. Use the rule you have been testing. Check by multiplying the two factors. Does it lead to a reasonable product."

$$\frac{\frac{3}{1} \times \frac{3}{2}}{1} = \frac{3}{1} \times \frac{3}{2} = \frac{9}{2} = 4\frac{1}{2}$$

Fred also uses a shorter form on the other 2 examples:

$$\frac{4}{7} \div \frac{1}{5} = \frac{\frac{4}{7} \times \frac{5}{1}}{\frac{1}{5} \times \frac{5}{1}} = \frac{\frac{4}{7} \times \frac{5}{1}}{1} = \frac{4}{7} \times \frac{5}{1} = \frac{20}{7} = 2\frac{6}{7}$$

$$\frac{3}{4} \div 6 = \frac{\frac{3}{4} \times \frac{1}{6}}{\frac{6}{1} \times \frac{1}{6}} = \frac{\frac{3}{4} \times \frac{1}{6}}{1} = \frac{3}{4} \times \frac{1}{6} = \frac{3}{24} = \frac{1}{8}$$

"Volunteers to test Fred's rule?"

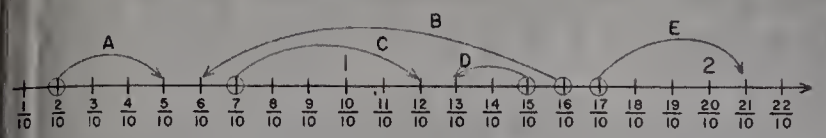
Alec Marson's rule

"Use Rule F on my machine."

Notes to accompany *Work Text* pages 68-73

Many children may already have learned to work with decimal fractions, adding, subtracting, multi-

Perhaps, on some inhabited planet, children first learn of fractions on a number-line such as the one below. Jumps are shown to suggest addition and subtraction. Try some examples.



$\frac{2}{10} + \frac{3}{10} = \frac{5}{10}$	D	$\frac{15}{10} - \frac{2}{10} = \frac{13}{10}$	G	$\frac{25}{10} - \frac{10}{10} = \frac{15}{10}$
$\frac{16}{10} - \frac{10}{10} = \frac{6}{10}$	E	$\frac{17}{10} + \frac{4}{10} = \frac{21}{10}$	H	$\frac{100}{10} + \frac{25}{10} = \frac{125}{10}$
$\frac{7}{10} + \frac{5}{10} = \frac{12}{10}$	F	$\frac{1}{10} + \frac{9}{10} = \frac{10}{10}$	I	$\frac{40}{10} - \frac{30}{10} = \frac{10}{10}$

You would not be surprised to learn that they developed a shorthand to save so much repetition of writing of 10 as denominator.

Can you discover their system?

.2 + .3 = .5	f	.1 + .9 = 1.0	k	32.5 − 5.0 = 27.5
1.6 − 1.0 = .6	g	2.5 − 1.0 = 1.5	l	.8 + 2.2 = 3.0
.7 + .5 = 1.2	h	10.0 + 2.5 = 12.5	m	10.2 + 3.9 = 14.1
1.5 − .2 = 1.3	i	4.0 − 3.0 = 1.0	n	8.4 + 8.4 = 16.8
1.7 + .4 = 2.1	j	11.0 − 9.5 = 1.5	o	10.3 − 8.8 = 1.5

Of course, they could write whole numbers in this same way because:

$= \frac{10}{10} = 1.0$	$2 = \frac{20}{10} = 2.0$	$3 = \frac{30}{10} = 3.0$	$4 = \frac{40}{10} = 4.0$
$= \frac{70}{10} = 7.0$	$9 = \frac{90}{10} = 9.0$	$10 = \frac{100}{10} = 10.0$	$11 = \frac{110}{10} = 11.0$

2 + .3 = 2.3	O	2 − .3 = 1.7	T	5 + 10.2 = 15.2
2 + 1.3 = 3.3	P	3.2 − 2 = 1.2	U	21.5 − 2 = 19.5
2 + 2 = 4.0	Q	5 − 3.4 = 1.6	V	8.1 − .9 = 7.2
.3 + 1.4 = 1.7	R	2 − 1.9 = .1	W	.8 + 10 = 10.8
1.6 + 1.0 = 2.6	S	8 − 3 = 5.0	X	7.7 − 7 = .7

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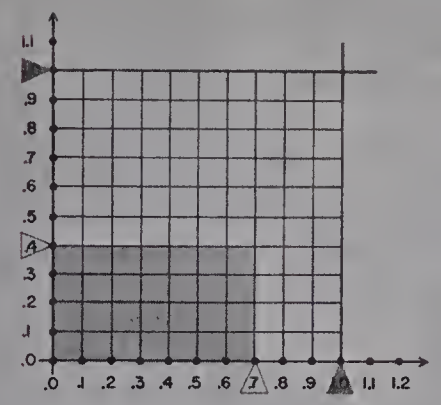
On this same planet, they would draw sketches to suggest the idea of multiplication and division. The diagrams might look like those below:



We know that $1 \times 1 = 1$

Also, we know that $\frac{10}{10} = 1.0 = 1$

And we know that $\frac{10}{10} \times \frac{10}{10} = \frac{100}{100} = 1$

The square made of heavy green and black lines is a diagram that suggests:

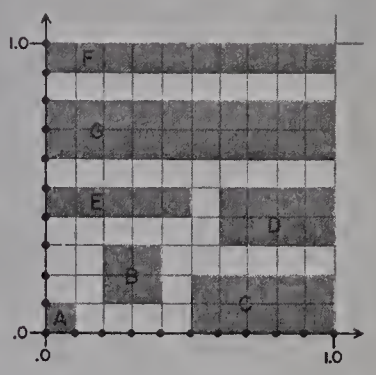
1×1 or $\frac{10}{10} \times \frac{10}{10}$ or 1.0×1.0 or $\frac{100}{100}$

$.4 \times .7$ or $\frac{4}{10} \times \frac{7}{10}$ or $\frac{28}{100}$

The area shaded blue suggests:

$.4 \times .7$ or $\frac{4}{10} \times \frac{7}{10}$ or $\frac{28}{100}$

In the diagram on the right, seven areas are shaded. What do they suggest?



Sketch:	Suggests that:
A.	$.1 \times .1 = \frac{1}{100}$
B.	$.2 \times .2 = \frac{4}{100}$
C.	$.5 \times .2 = \frac{10}{100}$
D.	$.4 \times .2 = \frac{8}{100}$
E.	$.5 \times .1 = \frac{5}{100}$
F.	$.1 \times 1.0 = \frac{10}{100}$
G.	$.2 \times .10 = \frac{20}{100}$

Instead of writing 100 so many times as a denominator, they would probably expand on their shorthand. Do you see their system?

$.1 \times .1 = .01$	$.1 \times 1.0 = .10$
$.5 \times .2 = .10$	$.80 + .27 = 1.07$

$.2 \times .4 = .08$	$1.0 \times 1.0 = 1.00$	$.1 + 1.0 = 1.1$	$.40 + .85 = 1.25$
$2.0 \times .2 = .40$	$1.0 \times 2.0 = 2.00$	$.01 + .09 = .10$	$4.1 + .5 = 4.6$
$.1 \times .5 = .05$	$1.5 \times 1.0 = 1.50$	$.01 + .1 = .11$	$4.1 + .05 = 4.15$

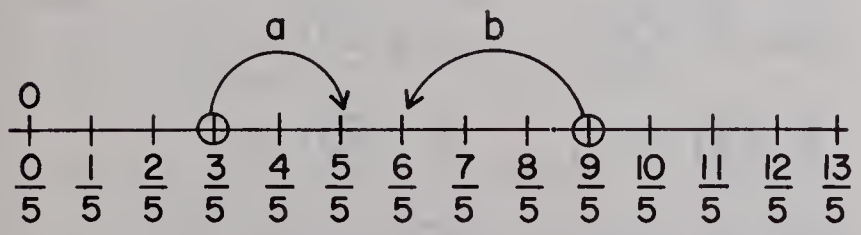
plying and dividing. In most cases, such children have learned certain rules that are reliable.

Other children may be only casually acquainted with decimal notation in other contexts than money.

The purpose of the next six *Work Text* pages—68 through 73—is to persuade all children to "think their way" through to sensible conclusions without relying on rules when working with decimal notations. A very simple story line can help set the tone.

A suggested development

Suppose, on another planet, children learned about fractions by considering several number lines. Here is one of them:



The ideas of addition and subtraction are suggested by jumps on the number line:

a. $\frac{3}{5} + \frac{2}{5} = \frac{5}{5}$
b. $\frac{9}{5} - \frac{3}{5} = \frac{6}{5}$

Now, they decide that they need a shorthand so they don't have to keep writing that "5" as denominator. They agree that when a numeral is circled, it implies "5th." So they would rewrite the examples above:

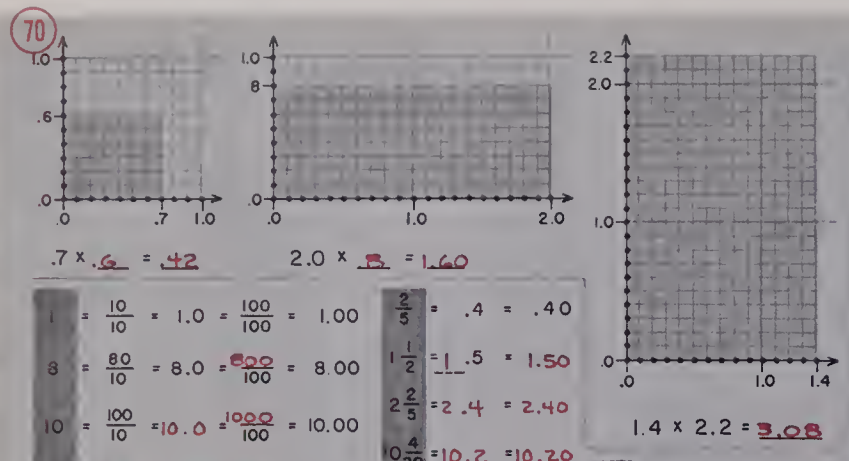
a. ③ + ② = ⑤

b. ⑨ − ③ = ⑥

How would they complete the following:?

c. ① + ⑨ = ___		f. ③ + ⑨ − ___ = ⑤	
d. ⑬ − ___ = ⑧		g. ___ + ⑦④ = ⑪①	
e. ㉕ + ㉕ = ___		h. ①②⑨ − ㉝ = ___	

Then they decided to adopt a rule. It was: "No numeral larger than 4 is to be circled." ⓪, ①, ②, ③, ④ were admitted, but not ⑤, ⑥, ⑦, etc.



.7 × .6 = .42 2.0 × .8 = 1.60 1.4 × 2.2 = 3.08

1	=	$\frac{10}{10}$	= 1.0	=	$\frac{100}{100}$	= 1.00	$\frac{2}{5}$	= .4	= .40	$\frac{3}{6}$	= $\frac{5}{10}$	= .5	= .50
8	=	$\frac{80}{10}$	= 8.0	=	$\frac{800}{100}$	= 8.00	$1\frac{1}{2}$	= 1 .5	= 1.50	$\frac{18}{12}$	= $\frac{15}{10}$	= 1.5	= 1.50
10	=	$\frac{100}{10}$	= 10.0	=	$\frac{1000}{100}$	= 10.00	$2\frac{2}{5}$	= 2 .4	= 2.40	$\frac{30}{4}$	= $\frac{75}{10}$	= 7.5	= 7.50
$\frac{1}{2}$	=	$\frac{5}{10}$	= 0.5	=	$\frac{50}{100}$	= 0.50	$10\frac{4}{20}$	= 10.2	= 10.20				
$\frac{1}{5}$	=	$\frac{2}{10}$	= 0.2	=	$\frac{20}{100}$	= 0.20	$\frac{28}{20}$	= $\frac{14}{10}$	= 1.40				
$\frac{9}{30}$	=	$\frac{3}{10}$	= 0.3	=	$\frac{30}{100}$	= 0.30	$\frac{15}{50}$	= $\frac{3}{10}$	= .30				
							15	= $\frac{150}{10}$	= 15.00				

	I				II				III			
a.	.3	+	.4	= .7	.07	+	.02	= .09	.3	+	.2	= .5
b.	.6	+	.4	= 1.0	.07	+	.06	= .13	.30	+	.20	= .50
c	1.0	−	.4	= .6	.25	+	.50	= .75	.3	+	.20	= .50
d	.8	+	.8	= 1.6	1.50	−	.55	= .95	.3	+	.05	= .35
e	2	×	.8	= 1.6	2	×	.75	= 1.50	.07	+	.03	= .10
f	1.6	÷	2	= .8	2	×	.50	= 1.00	.7	+	.03	= .73
g	1.6	÷	.8	= 2	3	×	.09	= .27	.30	−	.25	= .05
h	.9	−	.6	= .3	.18	÷	2	= .09	.4	−	.37	= .03
i	9.0	−	.6	= 8.4	.18	÷	.06	= 3	.80	−	.08	= .72
j	9.0	−	6.0	= 3.0	1.00	−	.10	= .90	.7	−	.07	= .63
k	1	×	.7	= .7	.01	+	.10	= .11	2.4	+	.12	= 2.52
l	10	×	.7	= 7.0	8	÷	100	= .08	1.8	−	.05	= 1.75
m	7	÷	10	= .7	8	−	200	= .04	.37	+	3.7	= 4.07

"Why do you think they made up such a rule?" A probable answer is, because ⑤, ⑥, ⑦, etc., are names for 1 or more.

$\frac{5}{5} = 1$ $\frac{6}{5} = 1\frac{1}{5}$ $\frac{7}{5} = 1\frac{2}{5}$

They decided that $\frac{5}{5}$ should be written, "1" or "1 ⓪." Here are more examples:

$\frac{6}{5} = 1①$ $\frac{9}{5} = 1④$

$\frac{7}{5} = 1②$ $\frac{10}{5} = 2$ or $2⓪$

They agreed upon certain rules for reading these new symbols. They simply used the same old language:

③ was read "three-fifths"
1 ① was read "one and one-fifth" or "six-fifths"
1 ⓪ was read "one" or "one and no fifths" or "five-fifths"
2 ⓪ was read "two" or "two and no fifths" or "ten-fifths"
② was read "two"

Read these symbols and expressions in all acceptable ways:

④	1 ③	2 ①
③ + ②	1 ② − ④	2 − 1 ②
2 ① + 1 ⓪	2 = 2 ⓪	1 ⓪ = 1 + ④

Let's try some examples in this shorthand:

②+②=④ ④+③+③=2 or 2⓪

③+②=1 or 1⓪ 1④-②=

④+④=1③ 1③-④=

1⓪+③= 3⓪-③=

1②+1①= 2-1②=

1③+1③= 3①-2=

They considered the following ideas involving expressions of fractions:

$2 \times \frac{1}{5} = \frac{2}{5}$ $\frac{1}{5} \times 2 = \frac{2}{5}$
$3 \times \frac{4}{5} = \frac{12}{5} = 2\frac{2}{5}$ $\frac{4}{5} \times 3 = \frac{12}{5} = 2\frac{2}{5}$

So, they write this in shorthand:

2 x ① = ② ① x 2 = ②

3 x ④ = 2② ④ x 3 = 2②

"Three times four-fifths equals two and two-fifths or twelve-fifths." Try these examples:

3 x ①=3 4 x③= 2 2

2x③= 1 1 4 x④= 3 1

②x 3 = 1 1 1② x 3 = 4 1

2 x 1②= 2 4 2⓪ x 2 = 4 or 4.0

and more:

$2 \times (2) = (4)$ $\quad 5 \times (1) = 1$ or 10

$2 \times (3) = 1(1)$ $\quad 2 \times ((2) + (1)) = 1(1)$

$(1) \times 4 = (4)$ $\quad (3) \times (1 + 3) = 2(2)$

$3 \times (3) = 2(1)$ $\quad 4 \times ((3) + (2)) = 4$ or 40

All this made good sense, but what about

$$(2) \times (4) = \_\_\_ ?$$

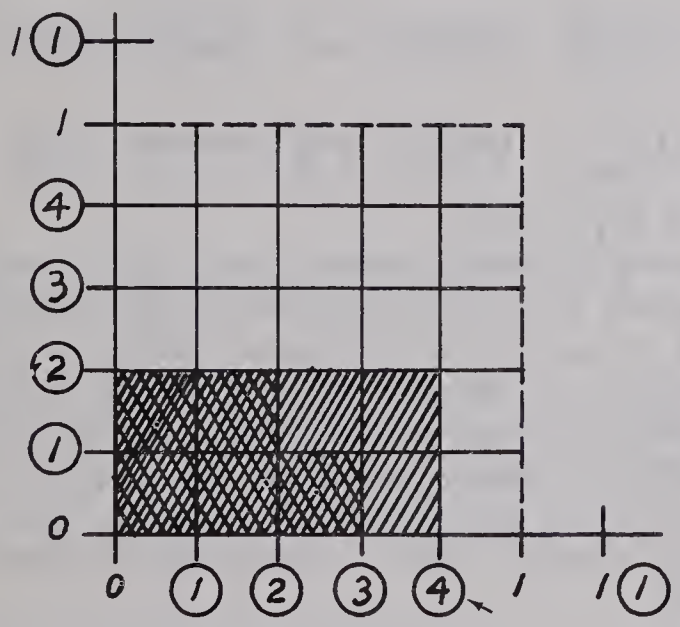
They understood that:

$$2 \times 4 = 8$$

$$2 \times (4) = 1(3)$$

$$(2) \times 4 = 1(3)$$

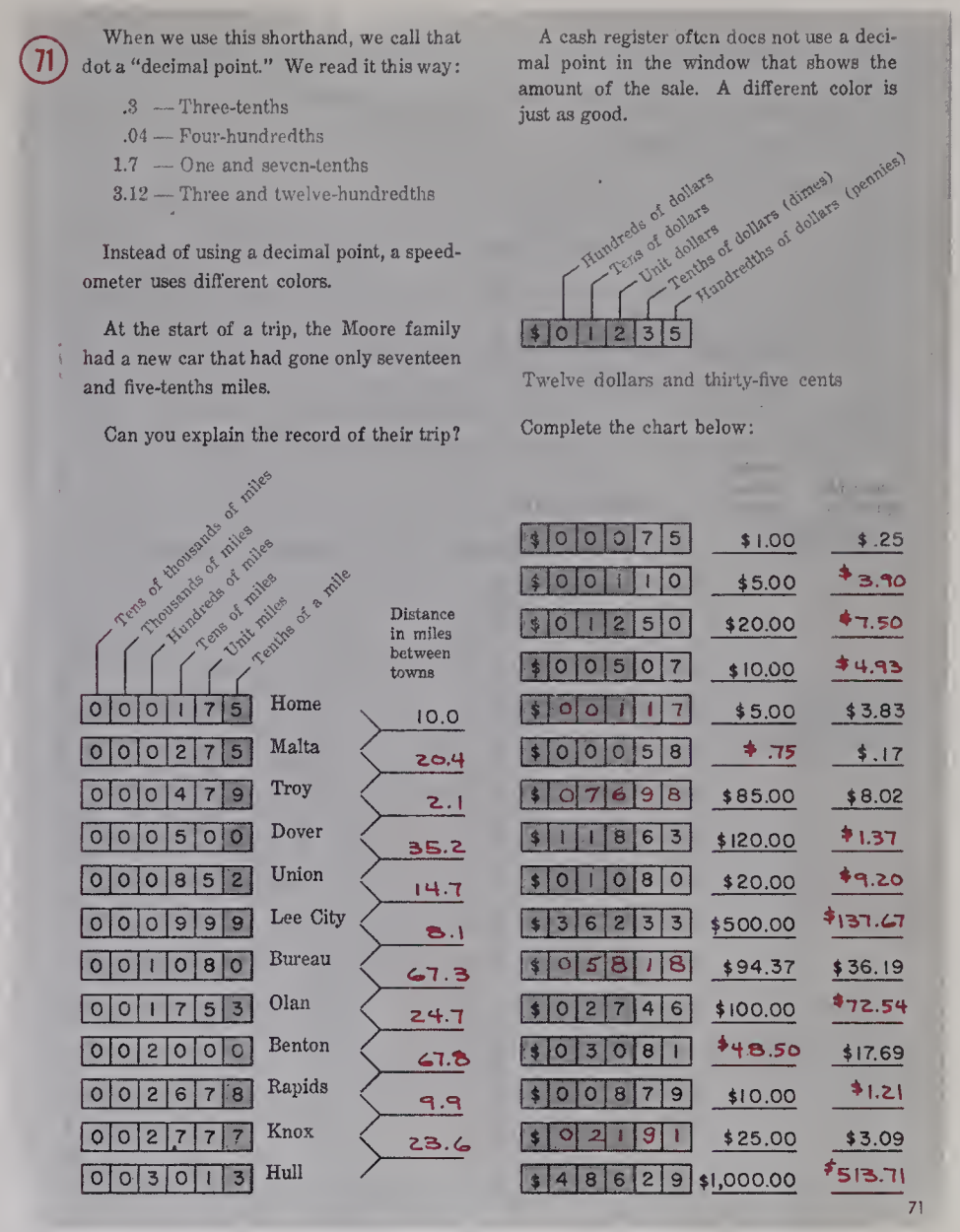
but ② × ④ was something else. So, they went back to their old way of writing ② and ④, and drew a sketch of the idea:



The sketch suggests several ideas. The heavy broken line and the area under it suggests that $1 \times 1 = 1$ and $1 \times 1 = {}^{25}/_{25}$, since there are 25 small sections in the picture that suggest 1×1.

Further, the shaded area suggests the idea of ② × ④ (shorthand for $^2/_5 \times {}^4/_5$) and that ② × ④ $= {}^8/_{25}$. The shaded area has two different shadings that suggest the idea that ${}^8/_{25} = {}^5/_{25} + {}^3/_{25}$ or $^1/_5 + {}^3/_{25}$.

Someone pointed out that $^1/_5 + {}^3/_{25}$ could be written ① $^8/_{25}$.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When we use this shorthand, we call that dot a "decimal point." We read it this way:
.3 — Three-tenths
.04 — Four-hundredths
1.7 — One and seven-tenths
3.12 — Three and twelve-hundredths
Instead of using a decimal point, a speedometer uses different colors.
At the start of a trip, the Moore family had a new car that had gone only seventeen and five-tenths miles.
Can you explain the record of their trip?
Tens of thousands of miles
Thousands of miles
Hundreds of miles
Tens of miles
Unit miles
Tenths of a mile
Distance in miles between towns
0 0 0 1 7 5 Home
0 0 0 2 7 5 Malta 10.0
0 0 0 4 7 9 Troy 20.4
0 0 0 5 0 0 Dover 2.1
0 0 0 8 5 2 Union 35.2
0 0 0 9 9 9 Lee City 14.7
0 0 1 0 8 0 Bureau 8.1
0 0 1 7 5 3 Olan 67.3
0 0 2 0 0 0 Benton 24.7
0 0 2 6 7 8 Rapids 67.8
0 0 2 7 7 7 Knox 9.9
0 0 3 0 1 3 Hull 23.6
A cash register often does not use a decimal point in the window that shows the amount of the sale. A different color is just as good.
Hundreds of dollars
Tens of dollars
Unit dollars
Tenths of dollars (dimes)
Hundredths of dollars (pennies)
$ 0 1 2 3 5
Twelve dollars and thirty-five cents
Complete the chart below:
$ 0 0 0 7 5 $1.00 $.25
$ 0 0 1 1 0 $5.00 $3.90
$ 0 1 2 5 0 $20.00 $7.50
$ 0 0 5 0 7 $10.00 $4.93
$ 0 0 1 1 7 $5.00 $3.83
$ 0 0 0 5 8 $.75 $.17
$ 0 7 6 9 8 $85.00 $8.02
$ 1 1 8 6 3 $120.00 $1.37
$ 0 1 0 8 0 $20.00 $9.20
$ 3 6 2 3 3 $500.00 $137.67
$ 0 5 8 1 8 $94.37 $36.19
$ 0 2 7 4 6 $100.00 $72.54
$ 0 3 0 8 1 $48.50 $17.69
$ 0 0 8 7 9 $10.00 $1.21
$ 0 2 1 9 1 $25.00 $3.09
$ 4 8 6 2 9 $1,000.00 $513.71
71

But the children decided that was too clumsy. Someone suggested that they simplify it this way:

$$(1)\frac{3}{25} = (1)(3)$$

to be read "one-fifth and 3 twenty-fifths" or "eight twenty-fifths."

Let's write other expressions with this idea in mind:

$$\frac{14}{25} = \frac{10}{25} + \frac{4}{25} = \frac{2}{5} + \frac{4}{25} = (2)(4)$$

$$\frac{19}{25} = \frac{15}{25} + \frac{4}{25} = \frac{3}{5} + \frac{4}{25} = (3)(4)$$

$$\frac{3}{25} = \frac{0}{25} + \frac{3}{25} = \frac{0}{5} + \frac{3}{25} = (0)(3)$$

$$\frac{32}{25} = 1 + \frac{7}{25} = 1 + \frac{5}{25} + \frac{2}{25} = 1 + \frac{1}{5} + \frac{2}{25} = 1(1)(2)$$

1 ① ② is read "one and one-fifth plus two twenty-fifths" or "one and seven twenty-fifths" or "thirty-two twenty-fifths."

MILES or HEADLINES — On blue lines, use no shorthand. On green lines, you may use shorthand.

APPROPRIATE PHRASES or STORIES

.1	One dime . . . Six minutes . . . (1/10 of a mile) Five-Hundred and twenty-eight feet. 360 seconds . . . 1/10 of a mile . . . 10¢ $.10 6 seconds 1.6 ounces
.5	Twelve hours . . . Fifty cents . . . Half-way . . . eight ounces half-pint 30 minutes . . . 1/2 off . . . 6 oranges 3 1/2 days in the middle.
1.7	One hour and forty-two minutes . . . Seventeen dimes . . . 170 pennies $1.70 . . . 102 seconds . . . 1.7 gallons . . . 1.7 miles 1.7 hours
.6 , .2	Six-tenths of a mile in two-tenths of an hour. In 2/10 of an hour we walked .6 of a mile.
.9 , .3 → ---	After we travelled nine-tenths of a mile, we still had three-tenths to go. That's one and two-tenth miles altogether. I had 9 dimes and spent 3 of them. I have 60¢ left. I put my 9 dimes in piles with 3 dimes in each. There were 3 piles.
1.6 , .4 → ---	One dollar and sixty cents is one dollar and twenty cents more than forty cents. One and 6/10 bushels together with .4 of a bushel make 2 bushels. In .4 of an hour I did as much as Bill did in 1.6 hours. I worked 4 times as fast.
.5 + .8 = 1.3	Half of a dime plus .8 of a dime is 13¢. One hour and 18 minutes is the same as 1/2 an hour plus .8 of an hour.
2 x 10.5 = 21.	Helen bought 2 dresses and paid $10.50 each. She spent $ 21.00. Uncle Bill lives 10 1/2 miles from our house. The round trip is 21 miles.

.25	Quarter of a mile. Half of a half-dollar. Fifteen minu Three eggs. Nine inches. Six hours. Three months.
2.50	Two and a half-dollars. Twenty-five dimes. Two and one- dimes. Thirty eggs. Two quarts and a pint.
1.30 , 2.8	He earned $1.30 in 2 hours and 48 minutes. In a minute and 18 seconds, the plane flew 2.8 miles. Before went to the store I had $2.80, but now I have only $1.30.
3.18 ÷ 2 = 1.59	On a round trip to Uncle Bills' I walked three and eig hundredths miles, so it must be one and fifty-nine-hundredths miles bet our houses. Two boys bought a small chemistry kit for $3.18 and sha the cost. Each boy paid $1.59.
3.24 10.30 + 6.68 20.22	The highway department made three road repair one required 3.24 tons of concrete, another 10.3 t and the third 6.68 tons. It required 20.22 tons in
4.32÷1.08 = 4 4 x 1.08 = 4.32	Harry spent $4.32 for books that cost $1.08 each. He m have bought 4 books. Each time Bill cut the neighbors lawn, he was paid $1.08. Du one month, he earned $4.32 in this way, so he must have cut th lawn 4 times.
.6 x 2 = △ and 5 x △ = 6.0	The school is .6 miles from my house, so I walk twice that for each day. When there are 5 school days in a week I walk 6 miles betwee home and school. If each shoe weighs .6 of a pound then a pa would weigh 1.2, and 5 pairs would weigh 6 pounds. Six-tenth 2 hours is 1 hour and 12 minutes. Five times as long is 6 hours.

Write other expressions in this shorthand:

$\frac{3}{5} \times \frac{4}{5} = \frac{12}{25} = \frac{10}{25} + \frac{2}{25} = \frac{2}{5} + \frac{2}{25} =$ (2)(2)

$\frac{6}{5} \times \frac{6}{5} = \frac{36}{25} = 1 + \frac{10}{25} + \frac{1}{25} = 1 + \frac{2}{5} + \frac{1}{25} =$ 1(2)(1)

$\frac{6}{5} \times \frac{5}{5} = \frac{30}{25} = 1 + \frac{5}{25} + \frac{0}{25} = 1 + \frac{1}{5} + \frac{0}{25} =$ 1(1)(0)

(4) x (3) = $\frac{12}{25}$ = (2) + (0)(2) = (2)(2)

(4) x (4) = $\frac{16}{25}$ = (3) + (0)(1) = (3)(1)

An idea was slipped in here that ought to be noted. How can we write $\frac{3}{25}$ or $\frac{1}{25}$ in this new shorthand?

(2) means "two fifths"

(1) means "one fifth"

But we can write $\frac{2}{25}$ as $\frac{0}{5} + \frac{2}{25}$ and $\frac{1}{25}$ and $\frac{0}{5} + \frac{1}{25}$.

(0)(2) means "two twenty-fifths"

(0)(1) means "one twenty-fifth"

Explore this question

A rule was introduced earlier to assure that $\frac{7}{5}$ would be written 1 (2) in this shorthand. The rule was stated in this way: "No number circled is to be greater than 4," so, only (0), (1), (2), (3), and (4) were acceptable. Can we extend the rule so that in the second circled number indicating "twenty-fifths" be limited to 4 or less? Can all fractions, $\frac{1}{25}$, $\frac{2}{25}$, $\frac{3}{25}$, etc., be written so that 4 is the largest numeral circled?

(0)	1
0	1
0	2
0	3
1	0
(1)	1

The answer is "Yes."

Now you can carry on this development as far as you like. Sometimes it is a little difficult to follow but, by referring to the equivalent common frac-

tions, trouble can be overcome. The difficulties encountered are the main point of the activity. Each of the difficulties has its parallel in decimal fraction notations—if one operates in decimal fractions with understanding.

Each aspect of this exercise helps build a clearer consciousness of the structure we are going to explore in the context of decimal fractions. In this useful pattern, the role of "fifths" will be played by "tenths," the role of "twenty-fifths" by "hundredths," etc.

The story line

The story line can be kept intact: "After learning the new shorthand for 'fifths,' the children begin with another number line—labeled in 'tenths.'" (See *Work Text* page 68.)

Of course they needed a new symbol so they could distinguish between $\frac{2}{5}$ and $\frac{7}{10}$, so they decided on the following method:

$\frac{2}{5} = (2)$ $\qquad$ $\frac{7}{10} = .7$

$1\frac{3}{5} = 1(3)$ $\qquad$ $2\frac{4}{10} = 2.4$

$2 = 2(0)$ $\qquad$ $5 = 5.0$

$\frac{2}{5} + \frac{1}{5} = (2) + (1) = (.3)$

$\frac{3}{10} + \frac{6}{10} = .3 + .6 = .9$

$\frac{3}{5} + \frac{4}{5} = (3) + (4) = 1(2)$

$\frac{7}{10} + \frac{8}{10} = .7 + .8 = 1.5$

And, you might experiment with:

$(2) = .4$ $\qquad$ $(1) + .7 = .9$

$.3 + .5 = (4)$ $\qquad$ $.7 - .3 = (2)$

perhaps considering whether these statements and others like them are true or false.

All the examples of *Work Text* page 68 lend themselves to the activities outlined above. Page 69 is a parallel of the investigation of $2 \times 4 = {}^{8}\!/_{25}$, and the agreement on symbols to express the result.

$(2) \times (4) = (0)(8)$

The sketches on page 69 support the conclusion that $^{7}\!/_{10} \times {}^{4}\!/_{10} = {}^{28}\!/_{10}$ and our new "tenths" shorthand is expanded to include "hundredths:"

$$.2 \times .4 = .08$$

$$\frac{28}{100} = \frac{20}{100} + \frac{8}{100} = \frac{2}{10} + \frac{8}{100} = .2 + .08 = .28$$

The examples on pages 69 and 70 are designed as exercises to explore a familiar idea from a fresh point of view. Each teacher must decide for himself the extent to which the story line suggested here is followed. Throughout, we approach questions in such a way that old rules don't help. There is no alternative for each child; he must think his way through, giving reasonable arguments to support each conclusion. For some children, this may be their first opportunity to study the structure of the system we use in expressing decimal fractions and in working with them. On page 71, we consider two familiar pieces of apparatus that are based on this structure—the mileage accumulator of a car speedometer and the display section of a cash register.

The speedometer is a dramatic example of the fact that we will never write any numeral more than 9 in any "place"—as, in the "fifth" system, we admitted no numerals larger than 4. Each place in a speedometer mileage recorder is occupied by a drum with numerals from 0 through 9—and that's all.

Of course, the cash register is also constructed so that those same numerals and no others will ever show in a given place.

Pages 72 and 73 employ the headline story technique to stimulate a search for examples from life in which decimal fractions are used.

Notes to accompany *Work Text* pages 74-75

Page 74 provides further experience in long division, with an emphasis on neat arrangement of the work. This algorism in particular is susceptible to errors due to lack of neatness. Further, it again reminds children of the relationship between multiplication and division: if $18 \div 3 = 6$, then $3 \times 6 = 18$.

Page 75 develops the idea of "remainders" in division. We begin with a game. This game might be in-

troduced by other forms of this kind of game that focus attention on regrouping. For example:

$$3\overline{)36} = 3\overline{)\underline{30}+\underline{6}} = 3\overline{)\underline{18}+\underline{12}+\underline{6}} = 3\overline{)\underline{48}-\underline{12}}$$

$$4\overline{)28} = 4\overline{)\ \ +\ \ } = 4\overline{)\ \ +\ \ +\ \ } = 4\overline{)\ \ -\ \ }$$

etc.

Introduce a rule: all numbers written above the division symbol ( ) must be *odd* numbers and *different*.

$$2\overline{)20} = 2\overline{)\ \ +\ \ } =$$

$$2\overline{)\ \ +\ \ -\ \ -\ \ }$$

Because of this rule, the following are not accepable:

$$\frac{5+5}{2)10+10}$$ (both the same)

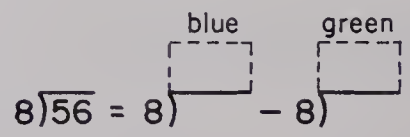
$$\frac{6+4}{2)12+8}$$ (not odd)

But the following meet both tests:

$$\frac{7+3}{2)14+6} \qquad \frac{9+\ 7-\ 5-1=10}{2)18+14-10-2}$$

This kind of game helps develop a sense of the freedom to regroup that is essential for the division algorism. The rules help develop ingenuity and lend the atmosphere of a puzzle to the activity.

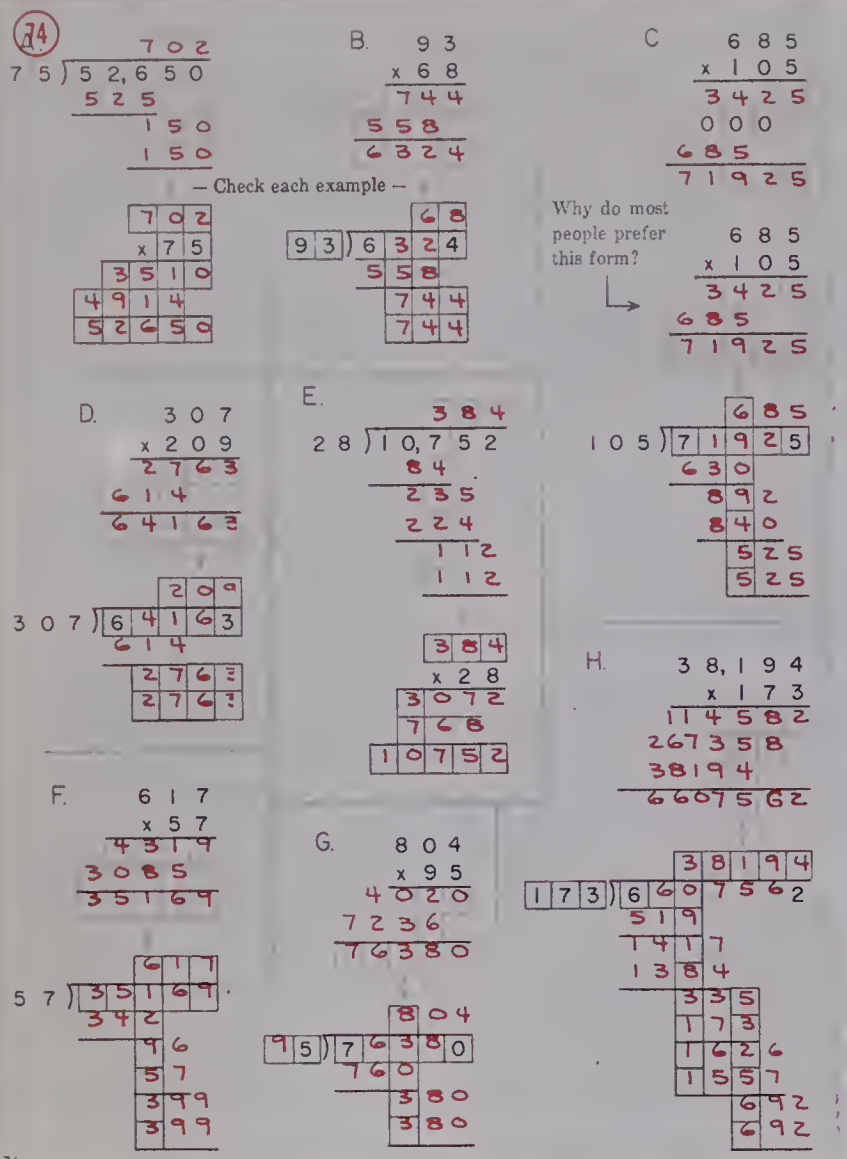
Eventually, the rules of the game are changed to those on page 75. When the first set of examples is completed, try a game with a slight twist:

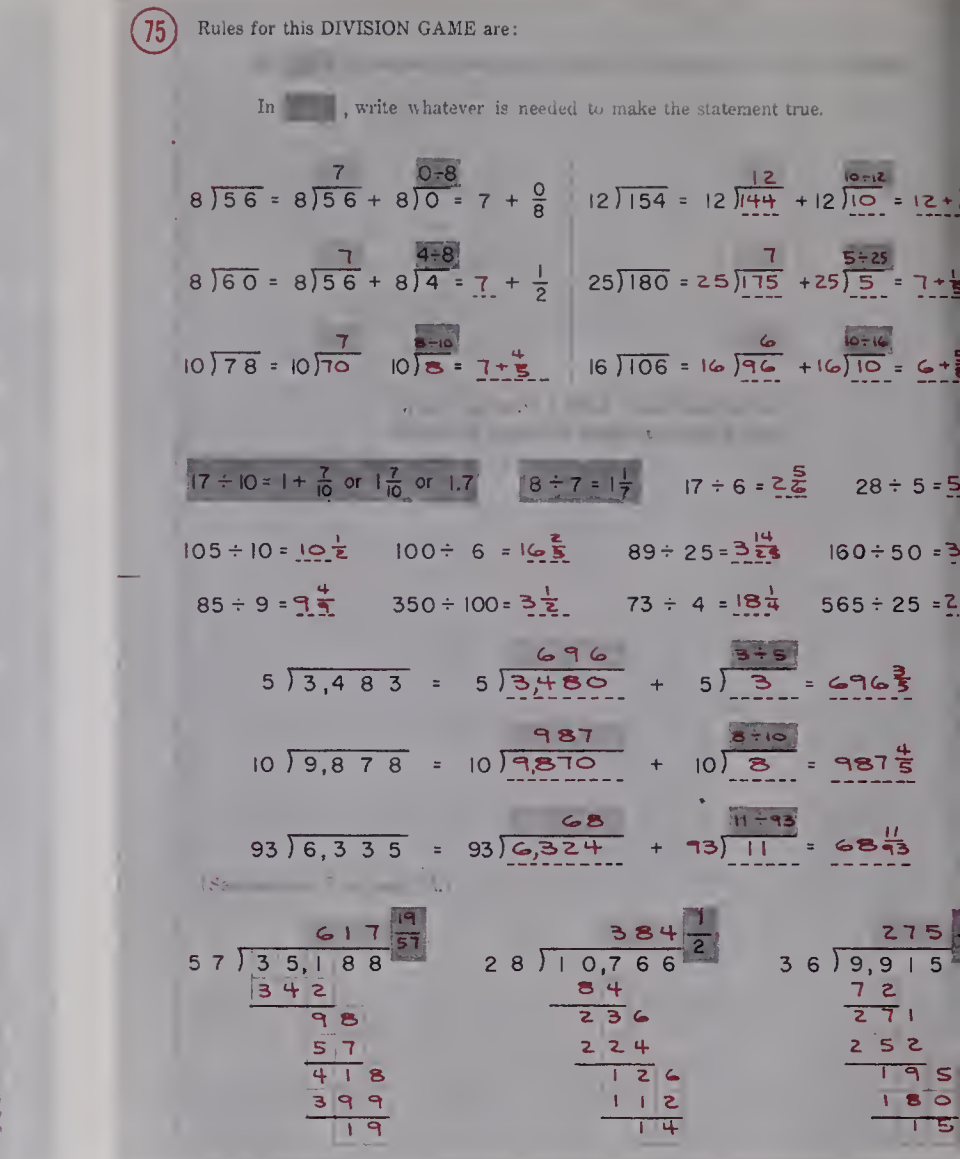


It is understood that blue is reserved for a whole number and green for a number that is less than one. The only acceptable response is:

$$8\overline{)60} = 8\overline{)64} - 8\overline{)\ 4} = 8 - \tfrac{1}{2} = 7\tfrac{1}{2}$$

$$10\overline{)78} = 10\overline{)80} - \overline{)2} =$$





$12\overline{)154} = 12\overline{)\quad} - 12\overline{)\quad} =$

$25\overline{)180} = 25\overline{)\quad} - 25\overline{)\quad} =$

Refer to *Work Text* pages 46, 47, 52, and 53 for other forms of division games.

Notes to accompany *Work Text* pages 76-79

Remainders are indicated in two different ways as shown on page 76. The example on the left is the more usual form, particularly when our main interest is in computation. However, when our chief concern is with the application of a result to a real-life situation, we choose the form which makes more sense in terms of that particular situation.

The idea of remainders has often been treated as just one of those details we have to handle in arithmetic. However, a study of remainders *can* be fascinating and illuminating. Pages 76, 77, 78 and 79 are trips into the land of remainders.

An ancient puzzle goes something like this: An old owl was perched in a tree above five barrels. His friend, the cat, came along, hungry and weary from a long trip.

"Say, friend cat, there's a mouse in one of the barrels that would be a nice supper for you."

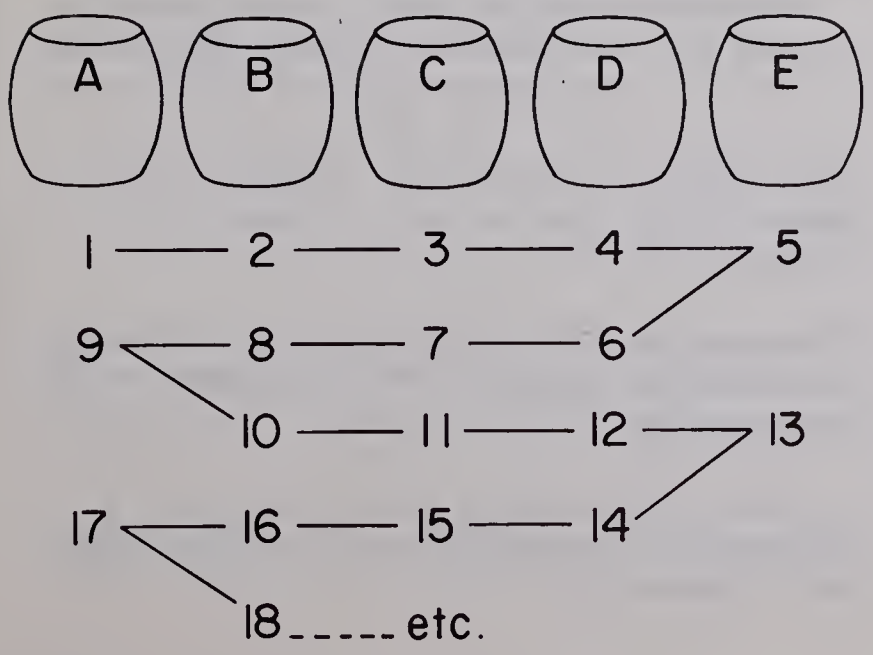
"Which one," the cat asked, "I see five barrels."

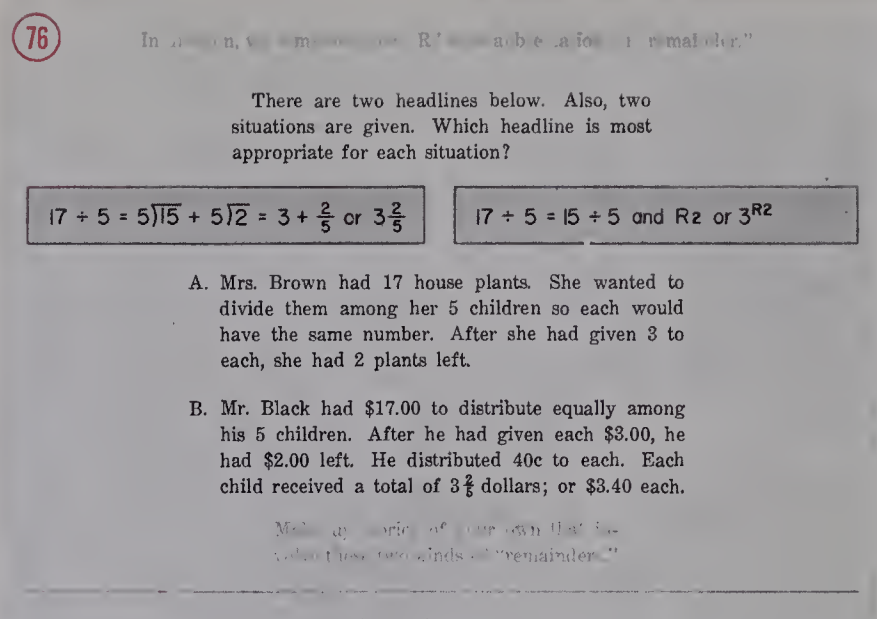
"Well, friend cat, it's in the 500th barrel."

"But there are only five barrels!" the cat protested.

"We have a peculiar way of counting here," the owl explained. "I'll show you how we count."

He thereupon drew this sketch:





76

There are two headlines below. Also, two situations are given. Which headline is most appropriate for each situation?

$17 \div 5 = 5\overline{)15} + 5\overline{)12} = 3 + \frac{2}{5}$ or $3\frac{2}{5}$	$17 \div 5 = 15 \div 5$ and R2 or 3^{R2}

A. Mrs. Brown had 17 house plants. She wanted to divide them among her 5 children so each would have the same number. After she had given 3 to each, she had 2 plants left.

B. Mr. Black had $17.00 to distribute equally among his 5 children. After he had given each $3.00, he had $2.00 left. He distributed 40c to each. Each child received a total of $3\frac{2}{5}$ dollars; or $3.40 each.

List numbers which —

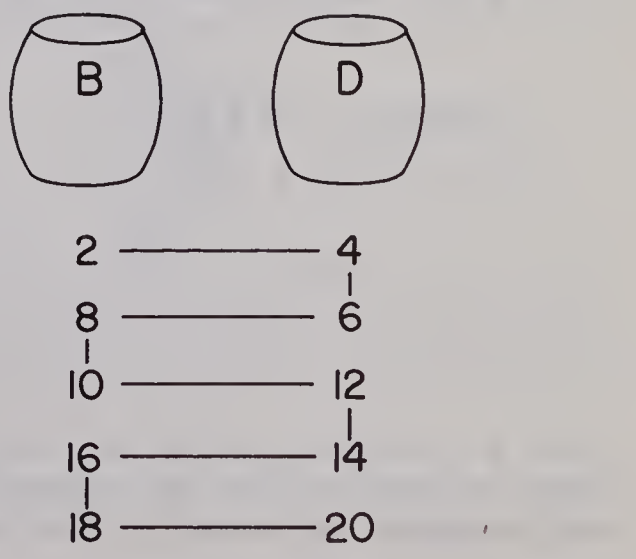
have remainders of	when divided by	a	b	c	d	e	f	g	h	i	j
2	5	17	32	2	7	12	22	27	37	102	1,977
4	5	4	9	14	64	84	104	19	99	169	8,769
3	4	23	7	3	11	43	103	123	87	91	1,003
1	3	1	4	7	10	13	16	19	301	34	22
1	2	1	3	5	7	9	11	15	17	19	7,121
1	6	1	7	19	37	43	61	97	601	145	355
4	9	22	49	67	4	13	22	31	40	94	364
6	7	20	76	13	6	13	20	27	34	62	699
8	11	30	19	52	74	118	96	129	283	602	1,108
5	8	69	101	5	13	21	29	37	45	85	1,669
7	10	37	17	67	7	27	47	57	197	1,777	9,687

76

"It's in the 500th barrel." And that's all the owl would tell his friend.

The cat was tired and hungry, and he couldn't wait to count all the way up to 500, so he put on his mathematical cap and, in a few moments, exclaimed, "I've got it!" And he was right. He selected the correct barrel without hesitation.

How did he accomplish the feat so quickly? He reasoned this way: It cannot be in barrels A or C or E, because it is obvious that each time I come to them, I will use an "odd number"—and 500 is not an odd number. So a shortcut would be to eliminate the odd numbers and count like this:



77

List numbers which satisfy both of two conditions:

have remainders of	when divided by	AND		(Take your time, but keep an eye open for patterns that help.) A	B	C	D	E	F	G
1	2	1	3	7	13	19	25	31	37	6,001
2	3	2	6	8	14	20	26	32	38	6,002
3	5	3	4	23	43	63	83	103	123	2,003
2	10	1	2	There are no such numbers.						
7	10	1	6	37	67	97	127	157	187	3,007
3	4	3	9	39	75	111	147	183	219	3,603
5	7	5	11	82	159	236	313	390	467	7,705

(Same rules as above) (Find a pattern, and then write examples as fast as you can.)

	R	AND when ÷	R	÷	H	I	J	K	L	M	N	O
I	1	2	1	5	1	11	21	31	41	51	101	1,971
II	2	3	2	4	2	14	26	38	50	62	182	1,202
III	4	6	4	7	4	46	88	130	172	214	256	8,404
IV	3	5	3	8	3	43	83	123	203	363	4,003	8,083
V	6	10	6	9	6	96	186	276	366	456	9,006	9,996
VI	1	3	1	5	1	16	31	46	61	151	301	6,031
VII	1	3	2	5	7	22	37	52	67	157	187	1,552
VIII	1	3	3	5	13	28	43	58	73	163	463	1,078
IX	1	3	4	5	4	19	34	49	79	139	454	7,519
X	1	4	1	7	1	29	57	85	113	281	561	5,601
XI	2	4	1	7	22	50	78	106	134	162	302	2,878
XII	3	4	1	7	15	43	71	99	127	295	575	8,443
XIII	1	9	2	4	10	46	82	118	154	190	262	3,682
XIV	3	9	2	4	30	66	102	138	390	426	462	7,266
XV	5	9	2	4	14	50	86	122	158	410	446	7,250

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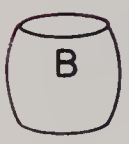
But that, also, would take too long. So, the cat decided to divide the two groups of numbers by 8 and study the results:



$8\overline{)2} = 0^{R2}$ $8\overline{)8} = 1^{R0}$ $8\overline{)10} = 1^{R2}$ $8\overline{)16} = 2^{R0}$ $8\overline{)18} = 2^{R2}$

$8\overline{)4} = 0^{R4}$ $8\overline{)6} = 0^{R6}$ $8\overline{)12} = 1^{R4}$ $8\overline{)14} = 1^{R6}$ $8\overline{)20} = 2^{R4}$

Finally, $8\overline{)500} = 62^{R4}$ (48, 20, 16)

So, he found the mouse in barrel D without further counting or delay. He noticed that all numbers associated with barrel B have remainders of 2 or 0 when divided by 8. He noticed that all numbers associated with barrel D have remainders of 4 or 6 when divided by 8. Since there is a remainder of 4 when 500 is divided by 8, the cat was sure 500 was associated with barrel D. (If the class is not convinced this is true, several of the pupils may perhaps they can extend it to 100 or 110 and find out if the remainder pattern is reliable that far—it may be more convincing when they see it at least that far.)

Another way to eliminate barrels A, C and E, as well as B, is to note the remainder pattern for each number associated with each barrel:

A:

When divided by 8 →	1	9	17	25	33	41	49
there is a remainder of →	1	1	1	1	1	1	1

B:

When divided by 8 →	2	8	10	16	18	24	26
there is a remainder of →	2	0	2	0	2	0	2

C:

When divided by 8 →	3	7	11	15	19	23	27
there is a remainder of →	3	7	3	7	3	7	3

D:

When divided by 8 →	4	6	12	14	20	22	28
there is a remainder of →	4	6	4	6	4	6	4

E:

When divided by 8 →	5	13	21	29	37	45	53
there is a remainder of →	5	5	5	5	5	5	5

In which group does 500 belong?

A remainder guessing game

"What is the smallest number that leaves a remainder of 1 when divided by 3 and when divided by 2?" The first suggested answer will probably be 7 because $2\overline{)7} = 3^{R1}$ and $3\overline{)7} = 2^{R1}$. However there is a smaller number that answers the question: $1 \div 2 = 0^{R1}$ and $1 \div 3 = 0^{R1}$.

We need to add a rule: no number less than the smaller divisor is allowed. So, the answer to the question is now 7.

"What is the smallest number that leaves a remainder of 2 when divided by 3 and when divided by 5?" Answer: 17.

A chart will help the game move ahead more rapidly:

	?	R	÷ by	÷ by
a.	7	1	2	3
b.	17	2	3	5
c.	21	1	3	7
d.	23	3	5	4
e.	11	1	5	2

A nice pattern is developed by these examples: Multiply the numbers in the columns on the right, and add the remainder to find the number

a. $(2 \times 3) + 1 \rightarrow 7$
b. $(3 \times 5) + 2 \rightarrow 17$
c. $(3 \times 7) + 1 \rightarrow 21$
d. $(5 \times 4) + 3 \rightarrow 23$
e. $(5 \times 2) + 1 \rightarrow 11$

But something happens in the next examples:

	?	R	÷ by	÷ by
f.	____	1	6	2
g.	____	3	4	6

In example *f*, $2 \times 6 + 1 = 13$ and $13 \div 2 = 6^{R1}$ and $13 \div 6 = 2^{R1}$; but $7 \div 2 = 3^{R1}$ and $7 \div 6 = 1^{R1}$; so, 7 is the correct answer.

In example *g*, $4 \times 6 + 3 = 27$ and $27 \div 4 = 6^{R3}$ and $27 \div 6 = 4^{R3}$; but $15 \div 4 = 3^{R3}$ and $15 \div 6 = 2^{R3}$; so, 5 is the correct answer.

Whenever the two divisors have a common factor, the answer is always the (product of the divisors) (common factor) + (remainder). The examples, *f* and *g*, can be computed this way:

f. $\frac{6 \times 2}{2} + 1 = 7$ g. $\frac{4 \times 6}{2} + 3 = 15$

(But, of course, you don't give this secret away.)

Change the game: "What is the smallest number that leaves a remainder of 3 when divided by 7 and 1 when divided by 4?" Answer: 10.

	?	÷ by	R	÷ by	R
h.	____	7 ,	3	4 ,	2
i.	____	5 ,	4	8 ,	7

That chart at *i* is shorthand for "What is the smallest number that, when divided by 5, leaves a remainder of 4, and when divided by 8 leaves a remainder of 7?" Answer: 39.

Let's study example *h*. List the numbers that comply with the first part of the statement: "... leave a remainder of 3 when divided by 7." They are: 3, 10, 17, 24, 31, etc. List those that leave a remainder of 2 when divided by 4: 2, 6, 10, 14, 18, 22, etc. Now, "What is the smallest number in both groups or sets of numbers?" Answer: 10.

Make a similar study of example *i*. List numbers that leave a remainder of 4 when divided by 5: 4, 9, 14, 19, 24, 29, 34, 39, 44, etc. List numbers that leave a remainder of 7 when divided by 8: 7, 15, 23, 31, 39, 47, 44, etc. Here is at least one trial and error method of being sure of finding the correct answer.

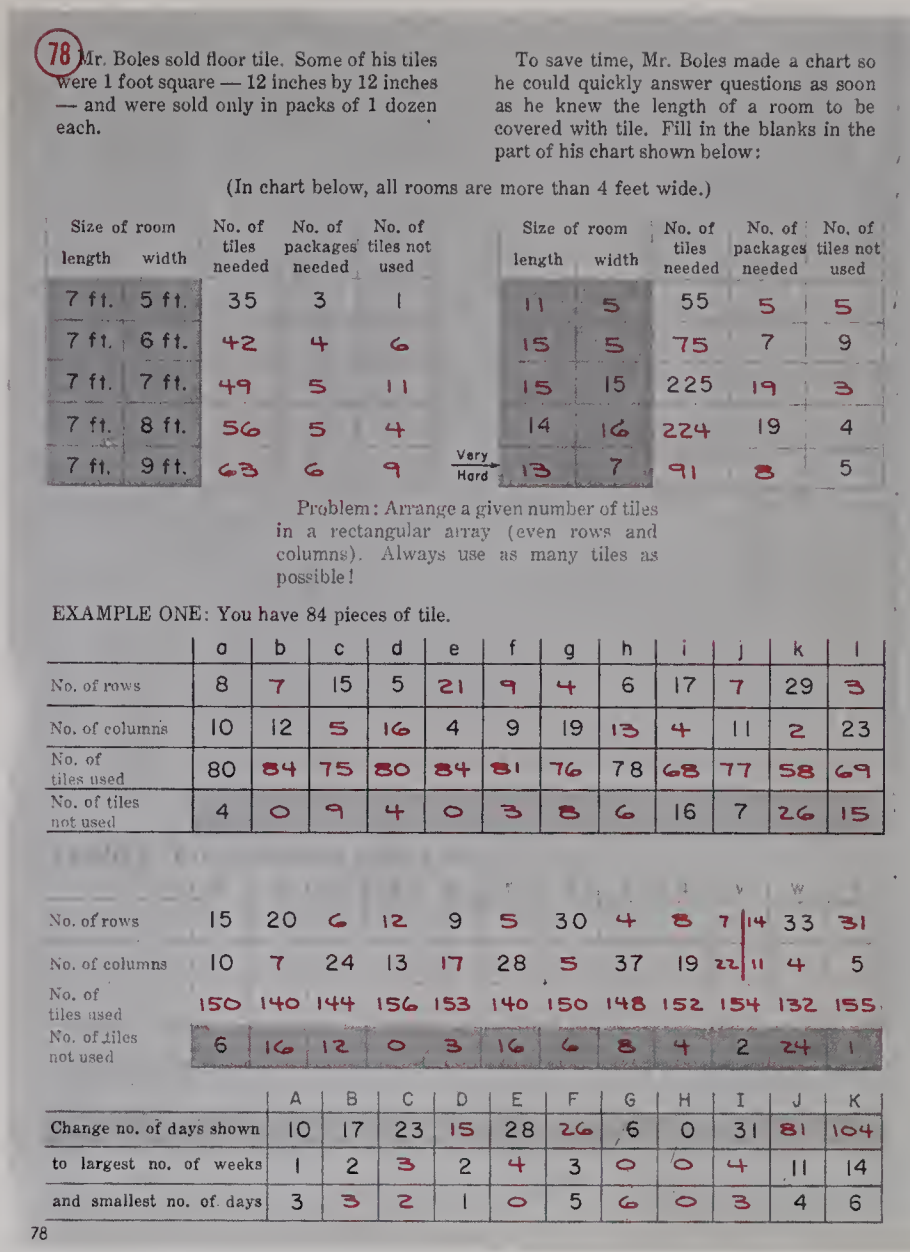
Change the rules. Same as before, but find the next smallest number that meets the tests. In example *h*, the trial and error chart would be:

3 , (10), 17 , 24 , 31 , 38 , 45 , 52 , 59 , 63
2 , 6 , (10), 14 , 18 , 22 , 26 , 30 , 34 , 38

Answer: 38.

So, the game becomes more difficult. Can you predict the next number that meets both tests? $10 + (4 \times 7) = 38$ and $38 + (4 \times 7) = 66$—but keep all rules a secret.

Another game.
Put a chart on the board something like this:


78

Mr. Boles sold floor tile. Some of his tiles were 1 foot square — 12 inches by 12 inches — and were sold only in packs of 1 dozen each.

To save time, Mr. Boles made a chart so he could quickly answer questions as soon as he knew the length of a room to be covered with tile. Fill in the blanks in the part of his chart shown below:

(In chart below, all rooms are more than 4 feet wide.)

Size of room length	Size of room width	No. of tiles needed	No. of packages needed	No. of tiles not used
7 ft.	5 ft.	35	3	1
7 ft.	6 ft.	42	4	6
7 ft.	7 ft.	49	5	11
7 ft.	8 ft.	56	5	4
7 ft.	9 ft.	63	6	9
11	5	55	5	5
15	5	75	7	9
15	15	225	19	3
14	16	224	19	4
Very Hard → 13	7	91	8	5

Problem: Arrange a given number of tiles in a rectangular array (even rows and columns). Always use as many tiles as possible!

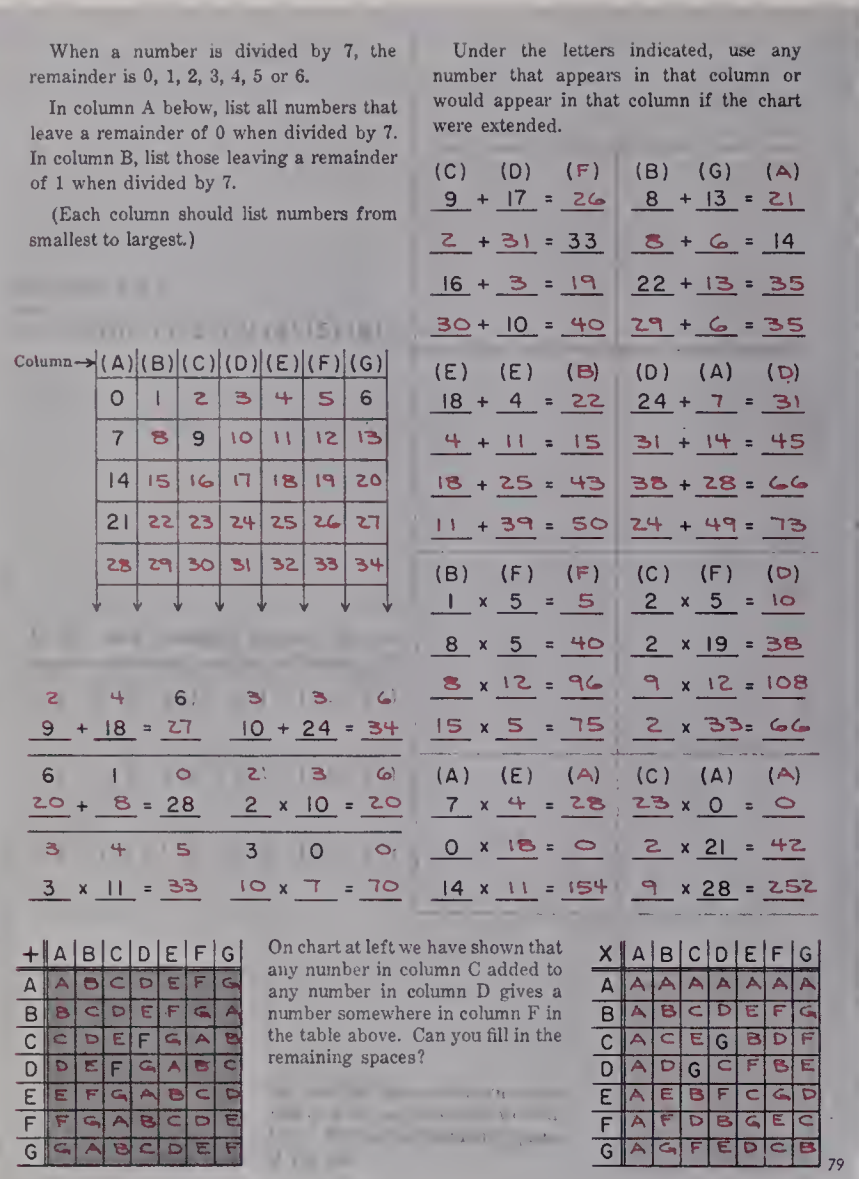
EXAMPLE ONE: You have 84 pieces of tile.

	a	b	c	d	e	f	g	h	i	j	k	l
No. of rows	8	7	15	5	21	9	4	6	17	7	29	3
No. of columns	10	12	5	16	4	9	19	13	4	11	2	23
No. of tiles used	80	84	75	80	84	81	76	78	68	77	58	69
No. of tiles not used	4	0	9	4	0	3	8	6	16	7	26	15

No. of rows	15	20	6	12	9	5	30	4	8	7	14	33	31
No. of columns	10	7	24	13	17	28	5	37	19	22	11	4	5
No. of tiles used	150	140	144	156	153	140	150	148	152	154	132	155	
No. of tiles not used	6	16	12	0	3	16	6	8	4	2	24	1	

	A	B	C	D	E	F	G	H	I	J	K
Change no. of days shown	10	17	23	15	28	26	6	0	31	81	104
to largest no. of weeks	1	2	3	2	4	3	0	0	4	11	14
and smallest no. of days	3	3	2	1	0	5	6	0	3	4	6

78


79
When a number is divided by 7, the remainder is 0, 1, 2, 3, 4, 5 or 6.
In column A below, list all numbers that leave a remainder of 0 when divided by 7. In column B, list those leaving a remainder of 1 when divided by 7.
(Each column should list numbers from smallest to largest.)
Under the letters indicated, use any number that appears in that column or would appear in that column if the chart were extended.
Column→ (A) (B) (C) (D) (E) (F) (G)
0 1 2 3 4 5 6
7 8 9 10 11 12 13
14 15 16 17 18 19 20
21 22 23 24 25 26 27
28 29 30 31 32 33 34
(C) (D) (F) (B) (G) (A)
9 + 17 = 26 8 + 13 = 21
2 + 31 = 33 8 + 6 = 14
16 + 3 = 19 22 + 13 = 35
30 + 10 = 40 29 + 6 = 35
(E) (E) (B) (D) (A) (D)
18 + 4 = 22 24 + 7 = 31
4 + 11 = 15 31 + 14 = 45
18 + 25 = 43 38 + 28 = 66
11 + 39 = 50 24 + 49 = 73
(B) (F) (F) (C) (F) (D)
1 x 5 = 5 2 x 5 = 10
8 x 5 = 40 2 x 19 = 38
8 x 12 = 96 9 x 12 = 108
15 x 5 = 75 2 x 33 = 66
2 4 6 3 3 6
9 + 18 = 27 10 + 24 = 34
6 1 0 2 3 6
20 + 8 = 28 2 x 10 = 20
3 4 5 3 0 0
3 x 11 = 33 10 x 7 = 70
(A) (E) (A) (C) (A) (A)
7 x 4 = 28 23 x 0 = 0
0 x 18 = 0 2 x 21 = 42
14 x 11 = 154 9 x 28 = 252
+ A B C D E F G
A A B C D E F G
B B C D E F G A
C C D E F G A B
D D E F G A B C
E E F G A B C D
F F G A B C D E
G G A B C D E F
On chart at left we have shown that any number in column C added to any number in column D gives a number somewhere in column F in the table above. Can you fill in the remaining spaces?
X A B C D E F G
A A A A A A A A
B A B C D E F G
C A C E G B D F
D A D G C F B E
E A E B F C G D
F A F D B G E C
G A G F E D C B
79

L_0	L_1	L_2	L_3	L_4
25	26	32	8	14
0				

These are "lists." In this game, all numbers that have a remainder of 3 when divided by 5 belong in L_3 (list 3); if remainder is 1 when divided by 5, it belongs in L_1 (list 1); etc.

A member of team A calls out a number "less than 100," and the next player on team B calls out the "list" it belongs in—indicating the remainder when that number is divided by 5. If the B player misses, he is eliminated (as in a spelling bee). If he succeeds, then a scorekeeper writes the number in the correct column, and the B player calls out a number less than 100, that is not already listed. If he repeats a number, he is eliminated.

The play alternates between A and B teams. (You may want to have a judge who eliminates those players who hesitate too long, either in calling out an answer or in calling out the name of a list.)

As the lists get longer, the game gets harder for the player who must call out a number. The game ends when each list contains 20 numbers. The winning team is, of course, the team with the fewest members eliminated.

These lists—L_0, L_1, L_2, L_3, and L_4—are now worth some study. You can play mind reader.

"Select two numbers less than 50 in L_3 (list 3). Add them together, but don't tell me your answer. You will find your answer in L_1.

"Select two numbers less than 50, one in L_3 and one in L_2. Add them together and you will find them in L_0."

Now, write your directions in shorthand:

$$L_3 + L_2 \rightarrow L_0$$

This means, "select a number less than 50 in L_3 and another number less than 50 from L_2. Add them. Look for your answer in L_0."

$$L_3 + L_3 \rightarrow L_1$$
$$L_1 + L_3 \rightarrow L_4$$
$$L_2 + L_2 \rightarrow L_4$$
$$L_0 + L_4 \rightarrow L_4$$
$$L_4 + L_4 \rightarrow L_3$$

Change to subtraction: "Select any number in L_3. From this, subtract any smaller number in L_1. Look for your answer in L_2."

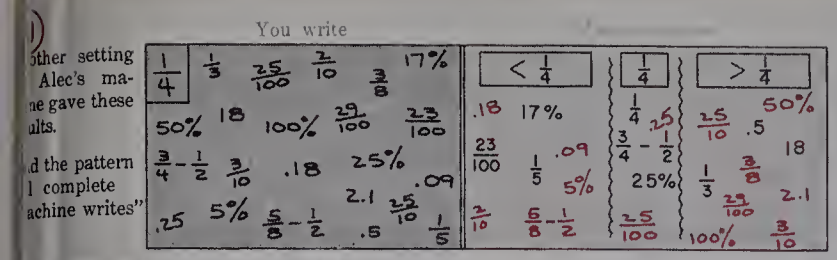
$$L_3 - L_1 \rightarrow L_2$$
$$L_3 - L_3 \rightarrow L_0$$
$$L_3 - L_4 \rightarrow L_4$$
$$L_0 - L_2 \rightarrow L_3$$
$$L_1 - L_3 \rightarrow L_3$$

Try mind reading with multiplication. "Select a number less than 10 in L_2 and another that is less than 10 in L_3. Look for the product of these numbers in L_1." Or, "Select any two numbers whose product is less than 100—select one factor from L_3 and the other from L_4. Look for your answer in L_2."

$$L_2 \times L_3 \rightarrow L_1$$
$$L_3 \times L_4 \rightarrow L_2$$
$$L_3 \times L_3 \rightarrow L_4$$
$$L_2 \times L_2 \rightarrow L_4$$
$$L_1 \times L_2 \rightarrow L_2$$
$$L_1 \times L_1 \rightarrow L_1$$

Each of these mind reading examples has its own pattern. Suggest that perhaps some pupils would like to study this matter and learn to be mind readers themselves.

These games have completely prepared the children for *Work Text* pages 76, 77, 78 and 79.



Here are some common uses of this new shorthand for an old idea:

25% off on all items.

We pay 3% on all savings deposits.

We won 50% of our games.

35% Dacron, 65% cotton.

100% Beef Frankfurters.

Pay only 75% of marked price.

What do you think these expressions mean?

→ 25% → 25% → 25%

$\frac{5}{0} = \frac{1}{4} = .25 = 25\% = \frac{2}{8} = \frac{5}{20}$ etc.

$\frac{0}{0} = \frac{1}{2} = .50 = .5 = 50\% = \frac{15}{30}$

$\frac{1}{5} = \frac{2}{10} = \frac{20}{100} = 20\% = .20 = \frac{3}{15}$

$\frac{3}{4} = \frac{30}{40} = \frac{75}{100} = 75\% = .75 = \frac{12}{16}$

Below, write sentences that include the following expressions:

50% 10% 4% 125%

I have read 50% of the book.

"10% off" is the same as "one tenth off." The cloth may shrink 4%. It usually takes me an hour to cut the lawn, but today it was wet, and it took me 125% of my usual time. 50% is 5 times as much as 10%. 4% of $2.50 is a dime.

Take the following true, using

= means "equals"
< means "less than"
> means "greater than"

$\frac{1}{2}$ = 50%	$\frac{1}{4}$ < 30%	10% = $\frac{10}{100}$
$\frac{1}{3}$ > 25%	$\frac{2}{5}$ = 40%	75% < .8
90% = $\frac{90}{100}$	100% = $\frac{100}{100}$	100% = 1
125% = $1\frac{1}{4}$	150% < $1\frac{3}{4}$	200% > $1\frac{1}{2}$
25% < $\frac{1}{4}$	$\frac{3}{4}$ > 72%	1% < .1
15% = $\frac{3}{20}$	$\frac{1}{4}$ < 30%	1% < $\frac{1}{2}$

81

Fill in the following as true statements. Do not repeat the same idea.

50% of 38 = 19	25% of 28 = 7	75% of 4 = 3
8 = 10% of 80	1 = 20% of 5	$2\frac{1}{2}$ = 5% of 50
25% x 36 = 9	80 = 200% x 40	75% x 1 = $\frac{3}{4}$
$2\frac{1}{4}$ = 15% x 15	150% x $\frac{1}{2}$ = .75	12 = 150% x 8
9 = 18% x 50	75 = 1% x 7,500	20% x 20 = 4

Each box below is to contain a part of the amount on the left. That part is indicated at the top of each column.

	50%	25%	10%	75%	5%	150%
1 hour	30 min.	15 min.	6 min.	45 min.	3 min.	90 min.
1 yard and 4 inches	20 in.	10 in.	4 in.	30 in.	2 in.	60 in.
5 dollars	$2.50	$1.25	$.50	$3.75	$.25	$7.50

	10%	20%	15%	110%	130%	250%
$25.00	$2.50	$5.00	$3.75	$27.50	$32.50	$62.50
2 minutes	12 sec.	24 sec.	18 sec.	132 sec.	156 sec.	5 min.
10 feet	1 ft.	2 ft.	18 in.	11 ft.	13 ft.	25 ft.

Write an appropriate story for each of the three headlines given below:

40% x 25 = 10

Marge missed 40% of the 25 problems. She had 10 wrong and 15 right. Henry saved 40% of his 25¢-a-week allowance. He saved 10¢ a week.

7 = 10% x 70

Because of a "10% off" sale, I saved 7¢ on the book. The regular price of the book was $.70. My goal is to save $70. I've saved $7., or 10% of this amount.

2.25 = 25 % of 9

After Bill Jones had gone $2\frac{1}{4}$ miles on his 9 mile trip, he was only $\frac{1}{4}$ of the way there—with $\frac{3}{4}$ to go. If 4 girls divided $9. evenly, each would get 25% of the total or $2.25.

81

Notes to accompany *Work Text* pages 80-82

Much mystery has sometimes been created about "per cent" in arithmetic. It has even become the land of "Three Cases." It has been a subject that has given rise to long sets of rules.

Actually, per cent is nothing more than an alternate shorthand for common fractions, and blood-brother to another alternate called decimal fractions. The simple fact is that:

$\frac{1}{4} = .25 = 25\%$

. . . three names for the same number.

To make any more of the matter is to lead children into confusion. As with common fractions, an expression such as 25% has many uses. When we speak of 25% of three dollars, we may be thinking about 25 out of each 100 cents and, since there are three-hundred cents, we refer to $3 \times 25¢$ or $75¢$. When we say that 25% of our class of 24 children is absent, we do not divide our class into 100 parts and count the absent parts, finding 25 missing. When we speak of 25% of $9284.84, we are talking about $9284.84 divided by 4.

All these ways of thinking have a common characteristic—all involve the idea of a number we can write as $\frac{1}{4}$ or .25 or 25% or $\frac{3}{12}$ or .250 or $\frac{2\frac{1}{2}}{10}$ or $\sqrt{\frac{1}{16}}$. It is the common characteristic we point to when we use any of these names.

As we discussed earlier in connection with multiplication of fractions, a translation problem arises. In common parlance, we say "twenty-five per cent of twenty-four"—and we write that as an expression of arithmetic in the following way: $25\% \times 24$, just as "one-fourth" of twenty-four is written: $\frac{1}{4} \times 24$.

"Of" in the language of everyday life (in the context we are discussing) becomes "times" in the language of arithmetic.

If we introduce "per cent" as another shorthand expression for common fractions, and translate "of" as the "times" of arithmetic, there is very little mystery—and a solid basis for understanding.

Work Text pages 80 and 81 are designed to help accomplish this goal. Page 80 uses an Alec Marson machine to introduce the idea that numbers have many names—that 25% is one of the names of $\frac{1}{4}$; that 17% is a number less than $\frac{1}{4}$; and that 100% is a number more than $\frac{1}{4}$. No distinction is made between 17%, .18 and $\frac{1}{3}$, because we are only inter-

82

Each group of numbers in the tinted area is written without order. Rearrange each group on the right as indicated by the signs.

a.	$\frac{1}{2}$	75%	.4	$\frac{8}{10}$	.49	41%	.4	< 41%	< .49	< $\frac{1}{2}$	< 75%	< $\frac{8}{10}$
b.	.7	125%	1	$\frac{16}{10}$	85%	1.5	$\frac{16}{10}$	> 1.5	> 125%	> 1	> 85%	> .7
c.	$\frac{1}{2}$	.03	3	.3	30	$\frac{1}{4}$	.03	< $\frac{1}{4}$	< .3	< $\frac{1}{2}$	< 3	< 30
d.	1.48	$\frac{9}{10}$	99%	150%	$1\frac{1}{100}$	$1\frac{1}{2}$	$1\frac{1}{2}$	= 150%	> 1.48	> $1\frac{1}{100}$	> 99%	> $\frac{9}{10}$
e.	$\frac{3}{10}$	20%	$\frac{1}{4}$	$\frac{3}{5}$	$\frac{1}{5}$	60%	$\frac{3}{5}$	= 60%	> $\frac{3}{10}$	> $\frac{1}{4}$	> $\frac{1}{5}$	= 20%
f.	1.00	$\frac{6}{3}$	200%	125%	1.9	$\frac{10}{8}$	1.00	< $\frac{10}{8}$	= 125%	< 1.9	< 200%	= $\frac{6}{3}$
g.	39%	$\frac{9}{20}$	$\frac{3}{100}$	.49	2%	$\frac{2}{5}$	2%	< $\frac{3}{100}$	< 39%	< $\frac{2}{5}$	< $\frac{9}{20}$	< .49
h.	$\frac{1}{8}$	1.0	6	$\frac{8}{2}$	$\frac{4}{36}$	12%	6	> $\frac{8}{2}$	> 1.0	> $\frac{1}{8}$	> 12%	> $\frac{4}{36}$
i.	$\frac{7}{20}$	.06	.35	06%	$\frac{3}{50}$	35.0%	$\frac{7}{20}$	= .35	= 35%	> .06	= $\frac{3}{50}$	= 06%
j.	.75	1000%	50%	10	$\frac{15}{20}$	$\frac{11}{22}$	50%	= $\frac{11}{22}$	< $\frac{15}{20}$	= .75	< 1000%	= 10
k.	$\frac{11}{100}$	$\frac{7}{50}$	13%	$\frac{3}{25}$	.1	9%	9%	< .1	< $\frac{11}{100}$	< $\frac{3}{25}$	< 13%	< $\frac{7}{50}$
l.	$\frac{1}{20}$	$\frac{1}{100}$	$\frac{5}{2}$	5%	3	250%	3	> 250%	= $\frac{5}{2}$	> $\frac{1}{20}$	= 5%	> $\frac{1}{100}$

Using <, > or =, make three true comments about each expression:

25% x 7	> 1	< 2	> 1.5	3 x $\frac{11}{10}$	> 2	< 7	< 10
10% x 83	> 8.00	< 80.0	< 800	3 x .2	< $\frac{6}{1}$	= $\frac{6}{10}$	> $\frac{6}{100}$
1% x 97	< 10	< 1	> .7	.3 x .2	< $\frac{6}{1}$	< $\frac{6}{10}$	= $\frac{6}{100}$
.10 x 25	> .20	> 2.0	< 20	.5 x .7	< 35	< 3.5	= .35
$\frac{10}{11}$ + $\frac{19}{20}$	> .5	< 2.5	> 1	8 x .4	< 32	= 3.2	> .32
.20 x 37	> .6	< 7	< 18	1.2 x 1.2	> 1	< 2	< 14.4
$\frac{1}{3}$ + $\frac{1}{2}$	> $\frac{1}{6}$	< 1	< 5	1.0 x 9	> .90	> .09	= 9.0
$\frac{7}{10}$ − $\frac{30}{100}$	> $\frac{1}{7}$	= .4	< 2.1	.3 x 25	= 7.5	> $\frac{3}{8}$	> 75%
7 ÷ 2	< 34	> .4	< 4	5 x .2	> 1%	> 10%	= 100%
.7 ÷ 2	> .3	< 5	< 1.4	.8 x .5	< 4.0	> .04	= .4
7 ÷ $\frac{1}{2}$	> 1.4	= 14	< 35	.3 x .3	< 1.0	< 0.1	> .01

82

ested in whether these numbers are less than, equal to, or more than ¼.

To reinforce this idea, write such expressions as: 17% of $5.00; .18 × $5.00; ⅓ of $5.00, and ask whether each refers to an amount of money that is less than, equal to, or more than ¼ of $5.00.

Then, "Which of the three expressions refers to the greatest amount of money? . . . to the least amount of money? What is the difference between the two least amounts? . . . the two larger amounts? . . . between the greatest and least amount? What is the sum of all amounts? How much is 17% + .18 + ⅓? How much is 68⅓% of $5.00?"

In this and similar activities, expressions of "per cent" take their rightful places as names for numbers, and "of" is translated easily into the "times" of arithmetic.

This later aim is the main theme of page 81. In the first set of examples, "of" and "x" start off on equal footing, but "x" soon wins out as the symbol preferred in mathematical sentences—though "of" is occasionally used in the headline story activities.

On page 81, the three headline story examples are deliberately chosen as examples of a difference that used to be singled out and called "Three cases of per cent." In the most often used form, these three headlines would look like this:

40% × 25 = ____
10% × ____ = 7
____% × 9 = 2.25

In terms that ought to be familiar by now, in the first example the factors are given and the product is missing; in the second, a factor is missing; and in the third, a factor is missing. The same can be said of the following:

3 × 4 = ____
3 × ____ = 12
____ × 4 = 12

We know that when a factor is missing, we often write the expression in terms of division, so the second and third headlines above can be written:

7 ÷ 10% = ____
2.25 ÷ 9 = ____%

And, in division form, 40% × 25 = ____ can be written:

____ ÷ 40% = 25
____ ÷ 25 = 40%

which is the same as changing the other set of expressions to:

12 ÷ 3 = ____ 12 ÷ ____ = 4
12 ÷ 4 = ____ ____ ÷ 4 = 3
12 ÷ ____ = 3 ____ ÷ 3 = 4

So, when we meet any problem such as

____% of 3,500 = 140 or
____% × 3,500 = 140

we can, if it helps keep the record straight, rewrite it as:

140 ÷ 3,500 = ____%

In the original expression, 140 is clearly the product and 3,500 a factor. A factor is missing. Since 140 ÷ 3,500 = .04, we know the full statement is 140 ÷ 3,500 = 4% or $\frac{4}{100}$ or $\frac{1}{25}$, etc.

Complete sentences below. (Do not simplify any of the fractions.)

$\frac{6}{10} \times \frac{8}{1} = \frac{48}{10}$ $\quad$ $\frac{2}{10} \times \frac{4}{10} = \frac{8}{100}$

$\frac{3}{10} \times \frac{9}{100} = \frac{27}{1,000}$ $\quad$ $\frac{5}{100} \times \frac{5}{100} = \frac{25}{10,000}$

$\frac{24}{100} \times \frac{2}{10} = \frac{48}{1,000}$ $\quad$ $\frac{8}{1} \times \frac{8}{10} = \frac{64}{10}$

$\frac{4}{00} \times \frac{12}{10} = \frac{48}{1,000}$ $\quad$ $\frac{2}{10} \times \frac{9}{10} = \frac{18}{100}$

$\frac{12}{100} \times \frac{9}{100} = \frac{108}{10,000}$ $\quad$ $\frac{12}{10} \times \frac{12}{1} = \frac{144}{10}$

$\frac{1}{10} \times \frac{13}{1} = \frac{13}{10}$ $\quad$ $\frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$

$\frac{7}{10} \times \frac{7}{100} = \frac{49}{1,000}$ $\quad$ $\frac{5}{10} \times \frac{7}{1} = \frac{35}{10}$

$\frac{12}{10} \times \frac{7}{100} = \frac{84}{1,000}$ $\quad$ $\frac{11}{10} \times \frac{5}{10} = \frac{55}{100}$

$\frac{4}{100} \times \frac{8}{100} = \frac{32}{10,000}$ $\quad$ $\frac{24}{100} \times \frac{57}{100} = \frac{1,368}{10,000}$

$\frac{1}{10} \times \frac{57}{100} = \frac{57}{1,000}$ $\quad$ $\frac{9}{100} \times \frac{2}{10} = \frac{18}{1,000}$

$\frac{13}{1} \times \frac{12}{10} = \frac{156}{10}$ $\quad$ $\frac{45}{10} \times \frac{57}{100} = \frac{2,565}{1,000}$

$\frac{24}{100} \times \frac{15}{10} = \frac{360}{1,000}$ $\quad$ $\frac{12}{100} \times \frac{18}{100} = \frac{216}{10,000}$

Record the results from each example on the left. Then complete the charts.

x	8	3	7	12
.6	4.8	1.8	4.2	7.2
.8	6.4	2.4	5.6	9.6
.5	4.0	1.5	3.5	6.0
1.0	8.0	3.0	7.0	12.0
1.2	9.6	3.6	8.4	14.4

x	.4	.6	.9	.5
.2	.08	.12	.18	.10
.7	.28	.42	.63	.35
.4	.16	.24	.36	.20
.9	.36	.54	.81	.45
1.1	.44	.66	.99	.55

x	.3	.8	1.2	.7
.09	.027	.072	.108	.063
.04	.012	.032	.048	.028
.11	.033	.088	.132	.077
.07	.021	.056	.084	.049
.03	.009	.024	.036	.021

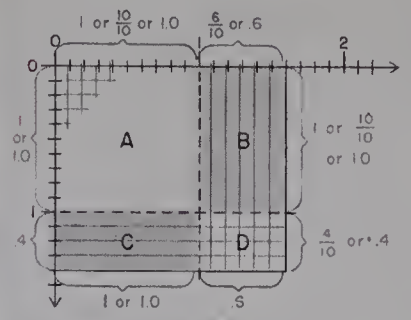
x	.2	13	1.5	.57	.18
24	.048	3.12	.360	.1368	.0432
.1	.02	1.3	.15	.057	.018
4.5	.90	58.5	6.75	2.565	.810
09	.018	1.17	.135	.0513	.0162
1.2	.24	15.6	1.80	.684	.216

x	.05	.09	.04	.06
.05	.0025	.0045	.0020	.0030
.10	.0050	.0090	.0040	.0060
.06	.0030	.0054	.0024	.0036
.08	.0040	.0072	.0032	.0048
.12	.0060	.0108	.0048	.0072

84

Familiar pictures can help explain the idea of multiplying decimal fractions.

"→" means "suggests that" or "is a picture that suggests that"



A → 1 x 1 = 1 and 1.0 x 1.0 = 1.00

B → 1 x .6 = .6 and 1.0 x .6 = .60

C → .4 x 1 = .4 and .4 x 1.0 = .40

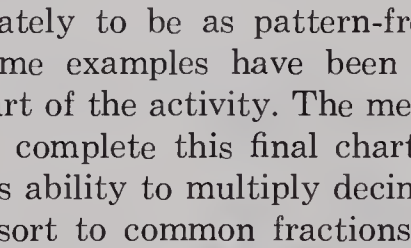
D → .4 x .6 = .24

A & B → (1 x 1) + (1 x .6) = 1.6

A & C → (1 x 1) + (.4 x 1) = 1.4

C & D → (.4 x 1) + (.4 x .6) = .64

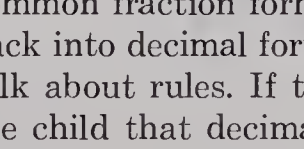
A & B & C & D → (1 x 1) + (1. x .6) + (.4 x 1) + (.4 x .6) = 2.24

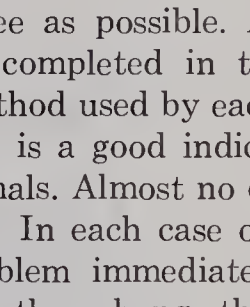


A & B → 1 x 1.6 = 1.6 or 16/10 or 1 6/10

and

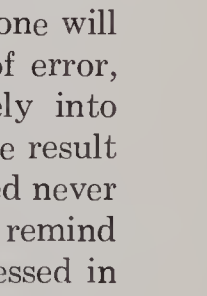
A & B → 1.0 x 1.6 = 1.60 or 160/100 or 1 60/100





A & C →
(1 x 1) + (.4 x 1) =
1 + .4 = 1.4 =
1.00 + .40 = 1.40

B & D →
1 x .6 + .4 x .6 =
.6 + .24 =
.60 + .24 = .84



C & D → .4 x 1 + .4 x .6 = .4 + .24 = .64

and

C & D → .4 x 1.6 = .64

These results look familiar. We already know that . . . . .

		1	6			1	4
x		1	4	x		1	6
		6	4			8	4
1	6				1	4	
2	2	4			2	2	4

C & D → .64
A & B → 1.60
A & B & C & D → 2.24
1.6 x 1.4 = 2.24

B & D → .84
A & C → 1.40
A & B & C & D → 2.24
1.4 x 1.6 = 2.24

84

Notes to accompany *Work Text* pages 83-85

The closest we come to mentioning *rules* applied to multiplication of decimals is the plan of work on page 83. The first part of the activity is concerned with the multiplication of common fractions, each of which has 1, 10 or 100 as denominator. Directions ask that none of the factors or products be simplified (or reduced).

The second part of the activity is to record each example in one of the five charts that fill the rest of the page, using decimal fraction notation rather than the common fraction form.

The final part of the activity is completion of each chart, using any means the student wants. He may convert each example into the common fraction form to find the result, and then record it in decimal fraction form. Or, he may already be able to multiply decimal fractions without any trouble. Or, he may see a pattern he accepts as a shortcut.

Because of the arrangement or selection of the examples encountered in each chart on the right, all results can be directly expressed in the same number of decimal "places." This fact obviously follows from the pattern in the groups of factors chosen for each chart.

Those not already familiar with this idea see it very quickly—and we hope that no discussion is necessary. If such discussion should prove necessary, postpone it until each child has completed the chart in the bottom left-hand corner, using whatever method he chooses.

In this last chart, the factors were chosen deliberately to be as pattern-free as possible. Already some examples have been completed in the first part of the activity. The method used by each child to complete this final chart is a good indicator of his ability to multiply decimals. Almost no one will resort to common fractions. In each case of error, however, translate the problem immediately into common fraction form—and then change the result back into decimal form. In this way, you need never talk about rules. If there is doubt or error, remind the child that decimals can always be expressed in common fraction form. Let each child make up his own rules or shortcuts as he proceeds. Any extra time required for this will be time saved later on.

Page 84 is an example of thinking through in several ways the problem presented by the expres-

sion: 1.4 × 1.6. We begin by drawing a sketch that helps suggest the idea—a pair of number lines that have a common point labeled 0. This diagram is one of the standbys that points to the row-column idea of multiplication.

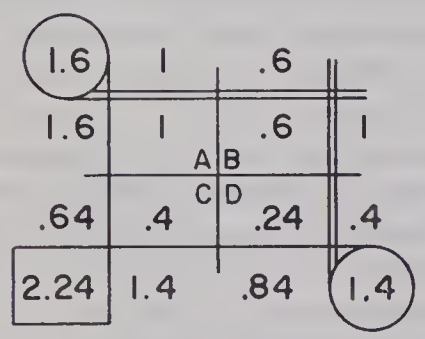
From 1.6 on one line and 1.4 on the other, lines are drawn to enclose an area we say suggests the idea of 1.4 × 1.6. Lines from the points labeled 1 divide the area into 4 parts. Each of these parts, labeled A, B, C and D, suggest less complex examples:

A → 1 × 1
B → .6 × 1
C → .4 × 1
D → .4 × .6

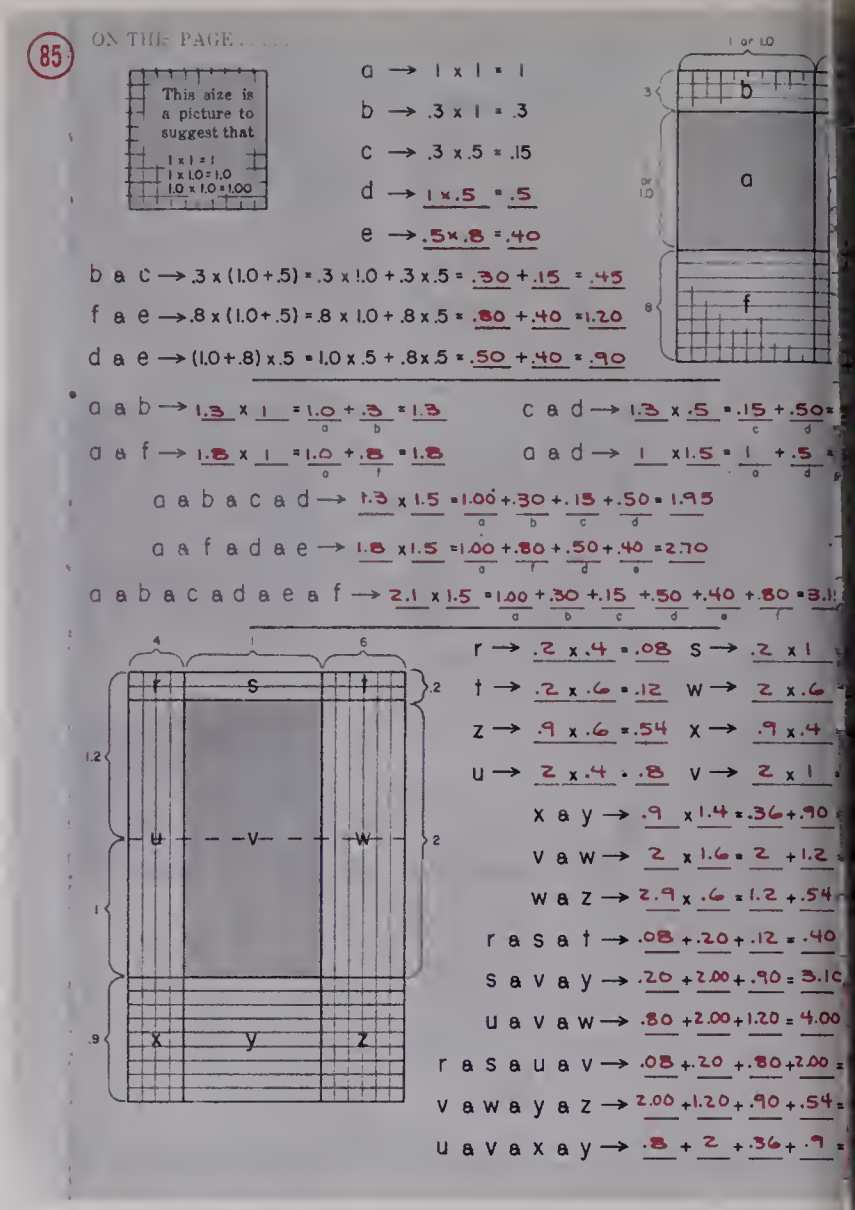
We can find the product in each case (using common fraction form, if necessary). By adding the results we feel sure we will find the number suggested by the whole area enclosed in the sketch—a numbr for which 1.4 × 1.6 is another name.

Next, we take a different approach. We analyze parts labeled A and B. We find a number that this combination suggests. We follow a like procedure with the combination of C and D. Finally, we add these two numbers—confident that the result will be the same as we found by the first method.

A third approach is similar. The combination of A and C is considered—and then the combination of B and D. Finally, the numbers found to be suggested by each combination are added—and we get an expected result. This is a pattern that is expressed in a form of the Cross Number Puzzle:



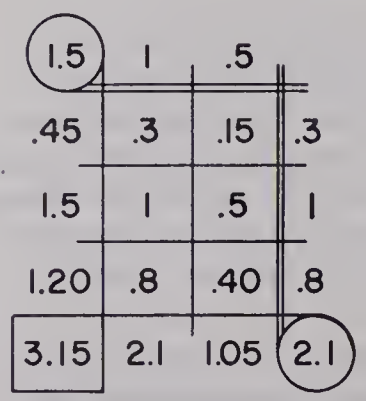
This amounts to nothing more than a record of the study on page 84. We considered the combination A and C, and the combination of B and D. The sum was noted in the check box. We also considered

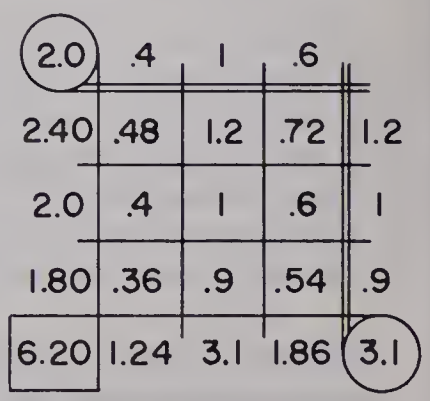


the combination of A and B; then C and D—which led us by another route to the same conclusion which is already noted in the check box.

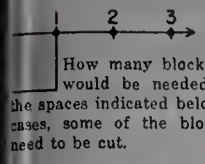
Finally, if we check the two examples shown in the blue-tinted area at the bottom of *Work Text* page 84, we find that exactly the same numbers turn up as partial and final products that we wrote as sums at the left of the rows and bottom of the columns.

A similar, though expanded, Cross Number Puzzle form can be used for each of the examples on *Work Text* page 85:

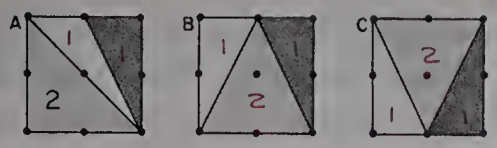


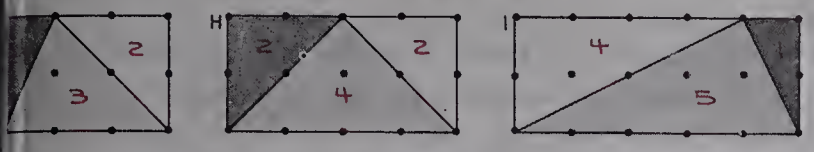


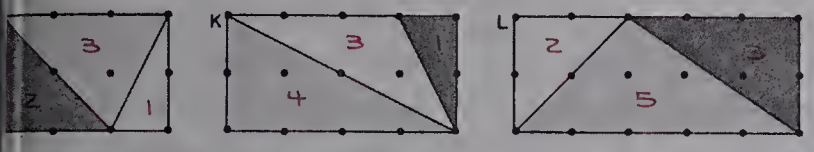
Indicate size of blue areas first.

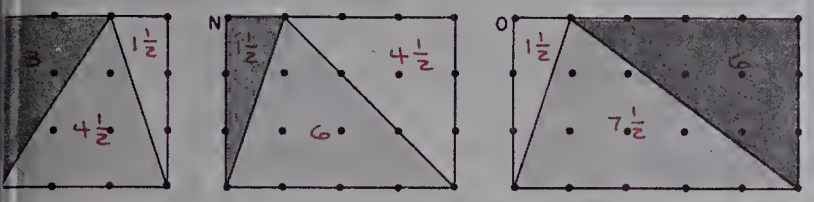


How many blocks this size would be needed to cover the spaces indicated below? In all cases, some of the blocks would need to be cut.

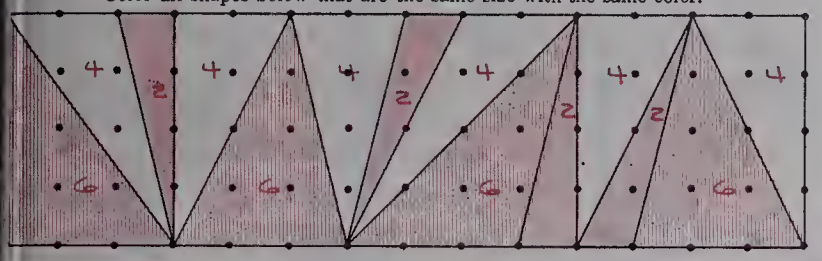








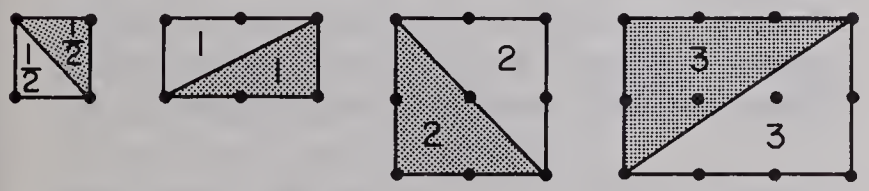
Color all shapes below that are the same size with the same color.



These diagrams, once the nature of the general problems are understood, will help children keep track of the results they find are needed as the work on page 85 develops further.

Notes to accompany *Work Text* pages 86-89

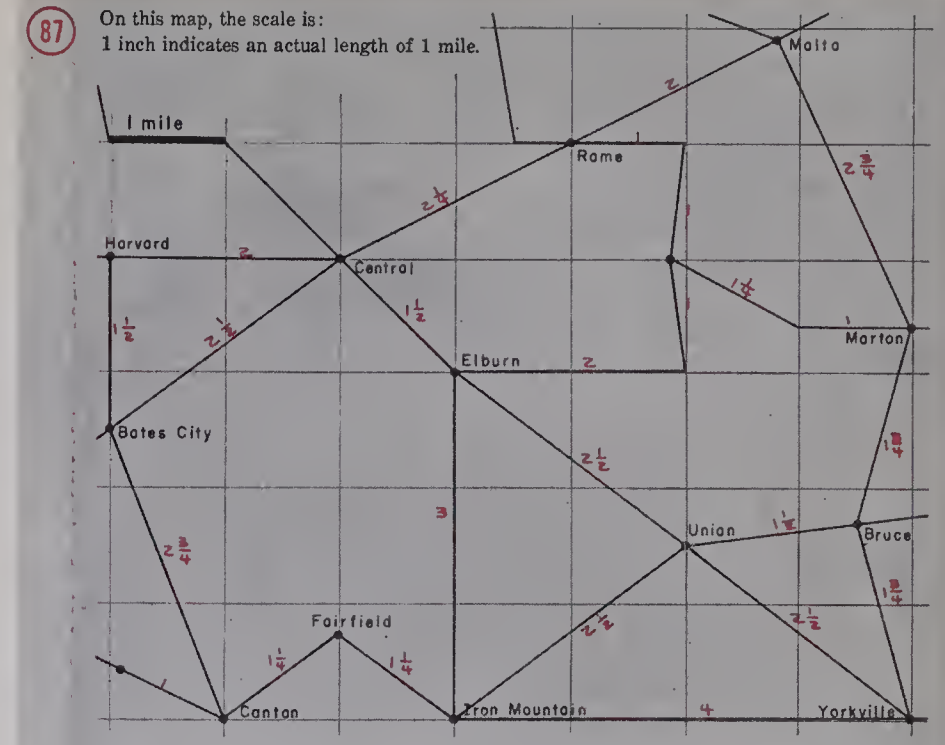
On the board, draw several squares and rectangles with one diagonal shown—and the area on one side of the diagonal shaded:



"Let's agree that this is a scale-drawing—that the distance between dots represents 1 foot. What is the area of each square or rectangle in numbers of square feet?

"In each diagram, which area indicated is larger —the shaded or the unshaded?" They, of course, represent the same size in each case.

"How large is the area represented by the shaded area in each square or rectangle?"



What is the shortest distance by road between the towns given below? (Give distance as the nearest quarter of a mile.)

Harvard and Rome: 4 1/4 miles
Bates City and Fairfield: 4 miles
Morton and Iron Mountain: 5 3/4 miles
Yorkville and Canton: 6 1/2 miles
Morton and Rome: 4 1/4 miles
Central and Yorkville: 6 1/2 miles
Harvard and Fairfield: 5 1/2 miles
Bates City and Malta: 6 3/4 miles
Elburn and Canton: 5 1/2 miles
Malta and Yorkville: 6 1/4 miles
Elburn and Morton: 5 1/4 miles

Mr. Drum is a salesman living in Bates City. He wants to leave home in the morning and call on customers in Fairfield, Central, Union and Bruce. What is the shortest distance he can travel, making all these stops and arriving at home in time for dinner? 17 1/4 miles

Find the length of the shortest trip by road from Morton to Rome by way of Fairfield. 15 miles

What is the shortest distance a crow would fly from Canton to Rome? 5 3/4 miles

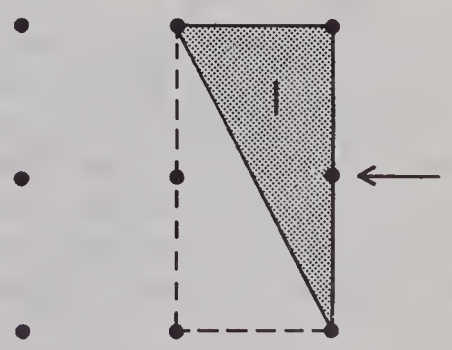
Which two cities shown are furthest apart as the crow flies? How far apart?
Harvard and Yorkville – 8 miles

87

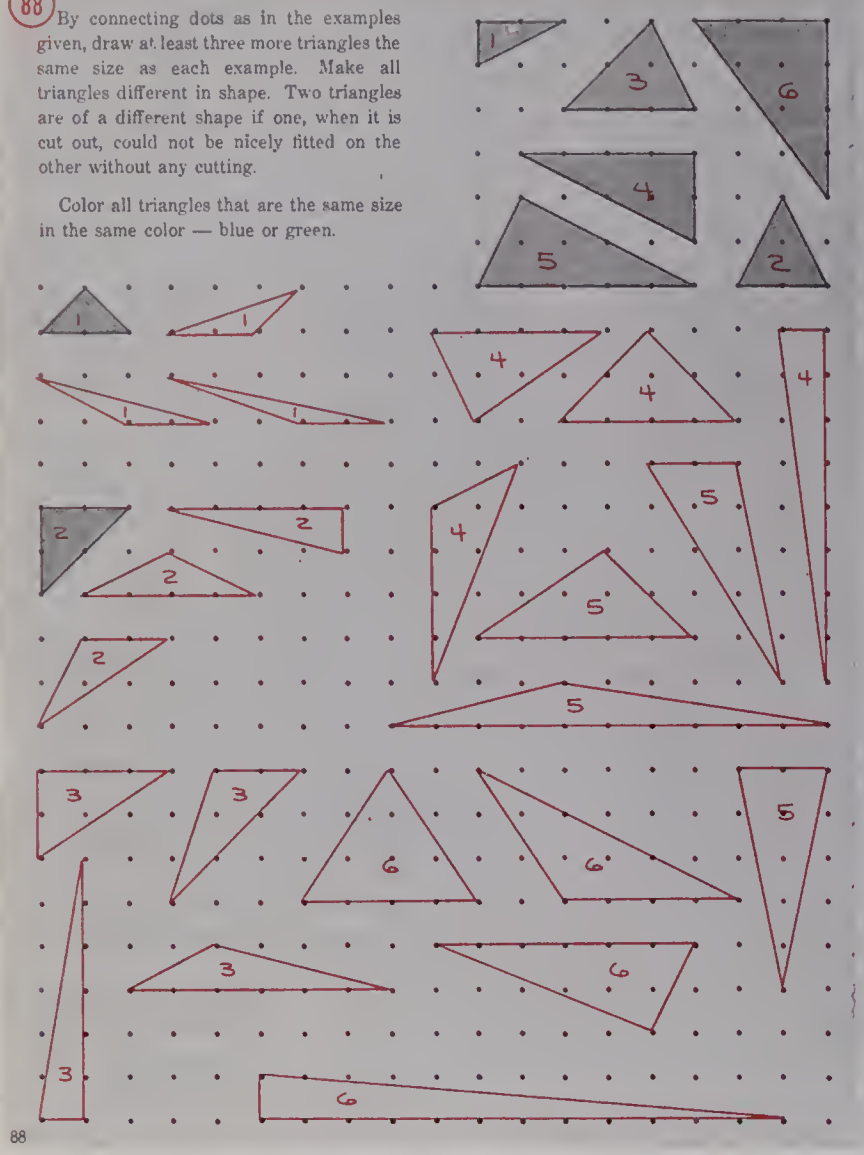
The understanding required to answer these questions is essential to the work on pages 86 and 88. Therefore, enough examples should be discussed to make sure all children can move ahead.

The agreement is that the square drawn by joining 4 dots as shown will be called "1 block." Then, sketches A, B and C each have an area of 4 blocks. "If sketch A is 4 blocks in size, how large is the area shaded blue?" This is the same question asked about the preliminary sketches on the board: it is half of 4, or 2 blocks in size.

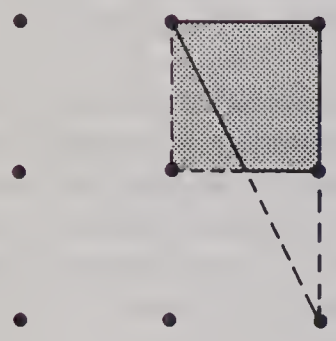
"How large is the area tinted green in A?" On the board, draw part of the sketch in A:



The green-tinted area can be seen as formed by a diagonal in a 2-block rectangle. So, it must be 1 block in area. In fact, if we cut it along the line

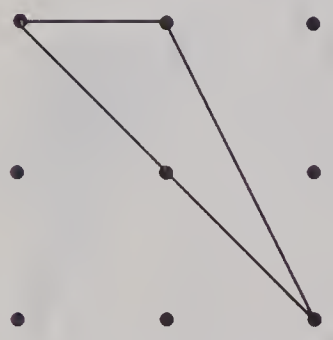

88
By connecting dots as in the examples given, draw at least three more triangles the same size as each example. Make all triangles different in shape. Two triangles are of a different shape if one, when it is cut out, could not be nicely fitted on the other without any cutting.
Color all triangles that are the same size in the same color — blue or green.
88

indicated by the arrow, the sharp point could be moved in this way:



Clearly, its area is 1 block.

"But what of the unshaded area in example A?"



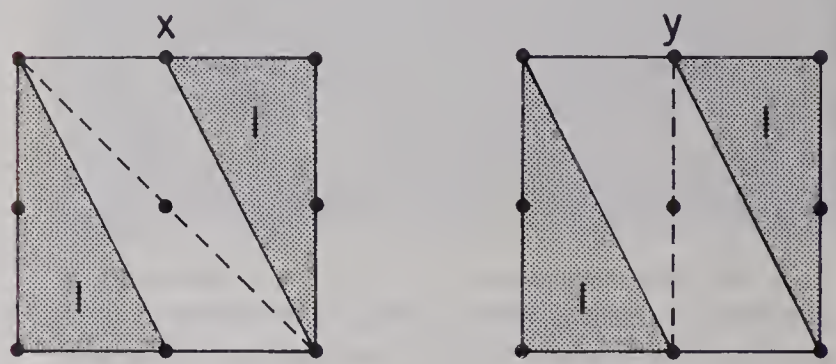
We argue our way to a conclusion in some such way as this: the unshaded area is what's left if you take the green- and blue-tinted areas out. The whole sketch is 4 blocks large. The blue-tinted area is 2, and the green-tinted area is 1—and together they are 3 blocks in size. Now, if 3 blocks are taken away from 4 blocks, 1 block would be left. So, the unshaded area must be 1 block in size.

Be sure that this kind of argument convinces everyone. If it doesn't, repeat it in another way: Together the unshaded and green-tinted areas are the same size as the blue area—which has an area of 2 blocks. Since the green area is 1 block in size, then the unshaded area must be the same size.

Don't hurry. The argument is sound, but the sketch doesn't show clearly that the smaller triangles are equal in area. Cut out a large scale drawing of this sketch. "Can you cut up the unshaded triangle and fit the pieces on the green one?" Not exactly, but with enough cuts it is a close match.

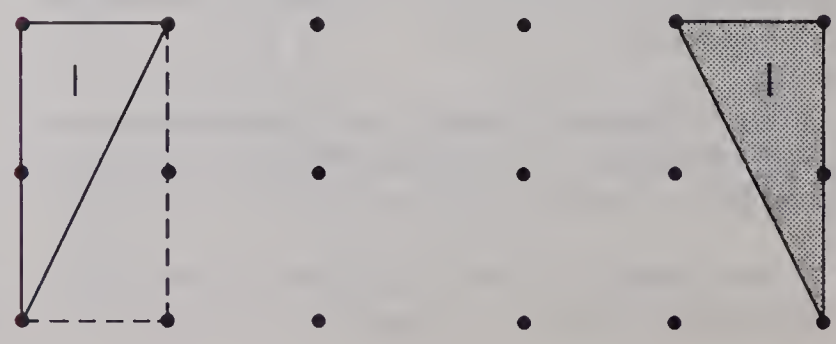
Another approach

Here are two sketches somewhat similar to A:



Each shaded area accounts for 1 of the 4 blocks—a total of 2. So, the unshaded area must account for 2. This is clear if the unshaded area is divided into two parts as in sketch *y*. "Does the broken line in sketch *x* divide the unshaded portion in two parts of equal size?" Now if you cut out a scale drawing, the two parts of the unshaded section will fit nicely —and help convince you they are the same size. And, since together they are the size of 2 blocks, each must be 1 block in size. Without real conviction that such arguments are reliable, the whole purpose of this work will be defeated. Keep cutting and fitting.

Consider sketch B. We can see the green-tinted and un-tinted sections as formed by diagonals of rectangles, 2 blocks in size:



Each is 1 block in size. Together, they account for 2 blocks of the 4 in the sketch. So, by our earlier line of argument, the blue-tinted area must be 2 blocks in size: $4 - 2 = 2$.

This time, we can cut to demonstrate. The green- and the un-tinted triangles will fit nicely on top of the blue triangle.

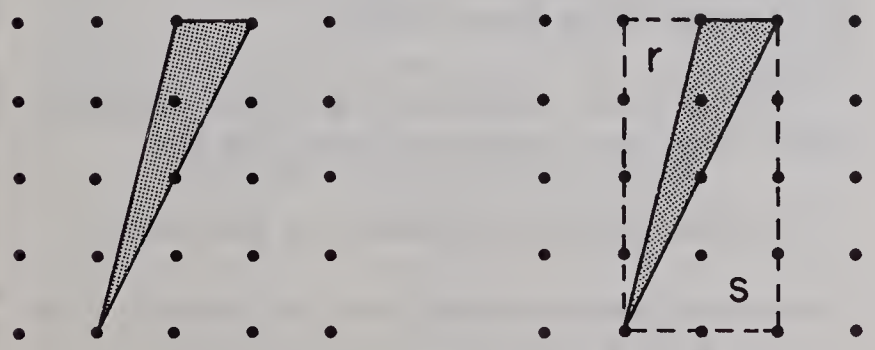
Sketch C is so similar to B that it needs no discussion here. Sketch D has much in common with example A. We easily find that the blue-tinted area accounts for 3 of the 6 blocks, and the green-tinted area for another block—a total of 4 out of 6. Then the un-tinted area must be 2 blocks in size. Certainly we could fit the two smaller areas on the blue area—each must be 3 blocks in size. If green is 1 block, then white must be 2 blocks in size. (While you are thoroughly convinced of this argument, give the children many opportunities to test it in one way or another.)

In each example through *o*, start the work by locating the two areas that are formed by diagonals of rectangles. The third area can always be found by the method outlined above.

Before tackling the last example on page 86, make a summary of the size of the blue-tinted areas along with the dimension of the sketch:

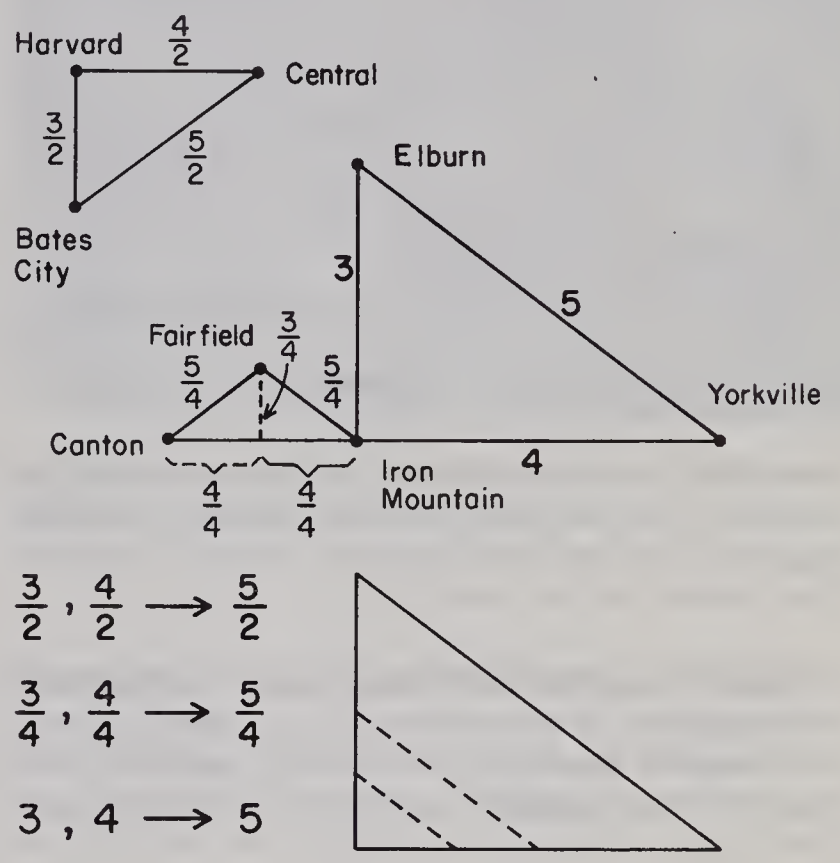
	Dimensions	Blue area in blocks
a.	2,2	2
b.	2,2	2
c.	2,2	2
d.	2,3	3
.	.	.
.	.	.
.	.	.
m.	3,3	4½
n.	3,4	6
o.	3,5	7½

In the final example, several of the areas need to be isolated, such as the following:



As shown on the left, the task is forbidding. But seen as part of a 2 by 4 rectangle, the mystery disappears. Of the 8 blocks in this rectangle, there are 2 in the area labeled *r* and 4 in the area labeled *s*—*r* and *s* together account for 6 of the 8 blocks. So, the area under investigation must have a size of 2 blocks. The most baffling shapes in this example will give up their sizes when approached in a similar fashion.

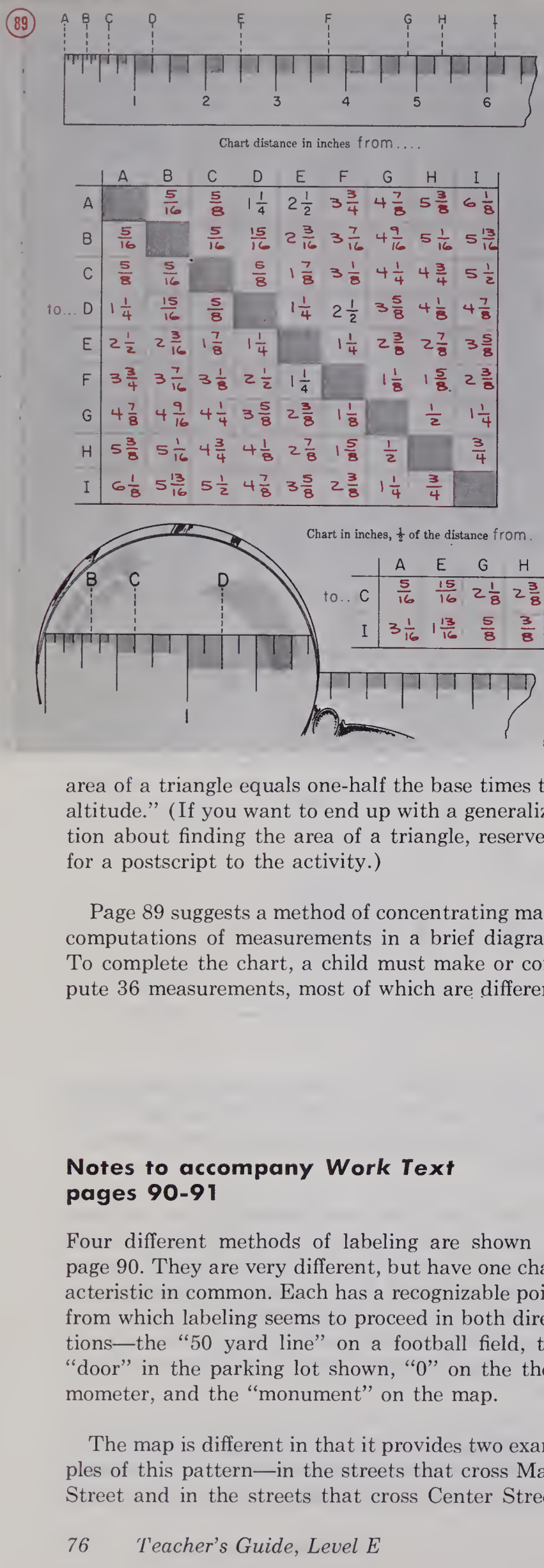
Work Text page 87 is simply an exercise in using a ruler, recording measurements to the nearest quarter inch, and adding common fractions. You might point out an interesting fact about some of the triangles formed by the roads. The distance can be expressed in a way that indicates a family resemblance:

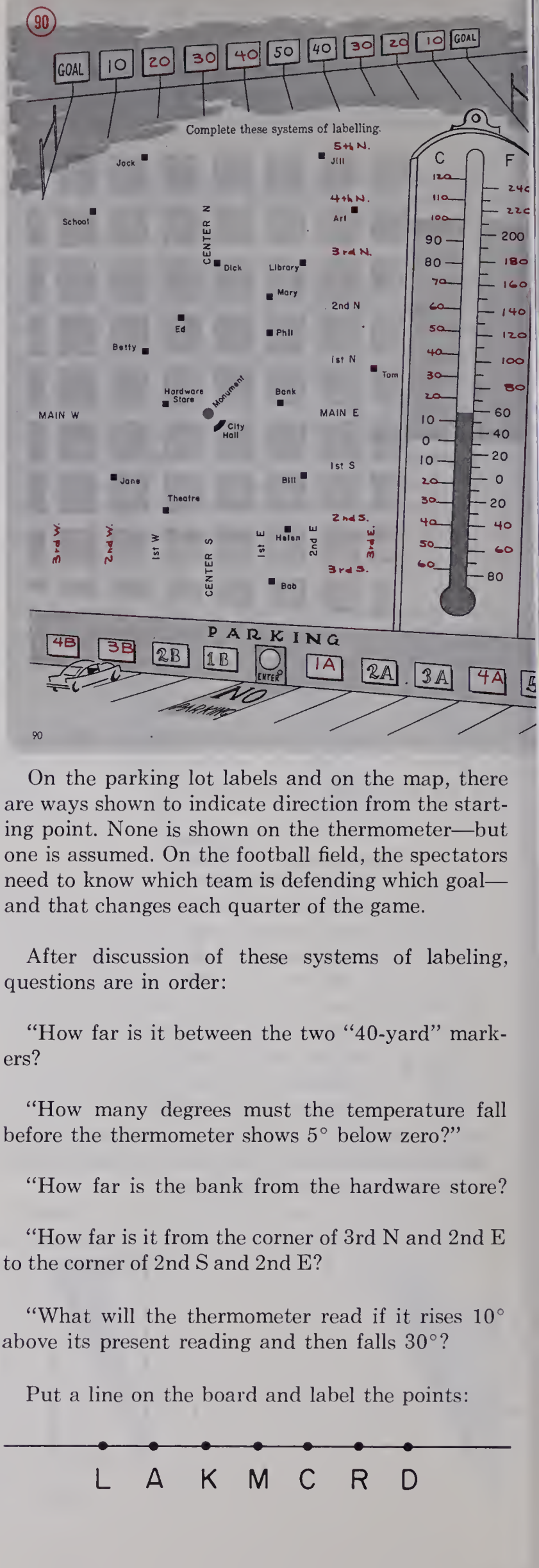


Someone might copy these triangles with tissue paper and see what happens if they are pasted one on top of each other. The family resemblance is obvious.

Page 88 will help reveal whether or not the idea on page 86 was grasped by each child. Let each individual work at the problems by himself, asking for help only when he is completely stymied. This will permit time to consider the work of each individual child. After the page is completed, have pairs of children exchange books to consider the examples used.

Finally, discuss some of the more interesting examples from sketches on the board. But, throughout the work, refrain from using any rule such as, "the





area of a triangle equals one-half the base times the altitude." (If you want to end up with a generalization about finding the area of a triangle, reserve it for a postscript to the activity.)

Page 89 suggests a method of concentrating many computations of measurements in a brief diagram. To complete the chart, a child must make or compute 36 measurements, most of which are different.

Notes to accompany *Work Text* pages 90-91

Four different methods of labeling are shown on page 90. They are very different, but have one characteristic in common. Each has a recognizable point from which labeling seems to proceed in both directions—the "50 yard line" on a football field, the "door" in the parking lot shown, "0" on the thermometer, and the "monument" on the map.

The map is different in that it provides two examples of this pattern—in the streets that cross Main Street and in the streets that cross Center Street.

On the parking lot labels and on the map, there are ways shown to indicate direction from the starting point. None is shown on the thermometer—but one is assumed. On the football field, the spectators need to know which team is defending which goal—and that changes each quarter of the game.

After discussion of these systems of labeling, questions are in order:

"How far is it between the two "40-yard" markers?

"How many degrees must the temperature fall before the thermometer shows 5° below zero?"

"How far is the bank from the hardware store?

"How far is it from the corner of 3rd N and 2nd E to the corner of 2nd S and 2nd E?

"What will the thermometer read if it rises 10° above its present reading and then falls 30°?

Put a line on the board and label the points:

L A K M C R D

"If we consider the distance between A and K as a unit, and all other similar distances as a unit, how far is it from K to C? D to M? A to R? C to L?

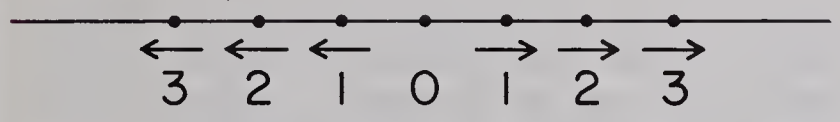
"Suppose Bill started at K, and Jim at R. How far apart are they? Each took a trip 1 unit in length. Now how far apart are they?" Answer: You haven't enough information. Which way did each go—right or left?

"What would this shorthand mean?"

	Bill starts	Jim starts	Distance apart
a.	← 1	→ 1	5
b.	→ 1	→ 1	*3*
c.	→ 1	← 1	*1*
d.	← 1	← 1	*3*

"At *a* it probably means that Bill started at K and went one unit to the left; Jim started at R and went one unit to the right. After these trips they were 5 units apart."

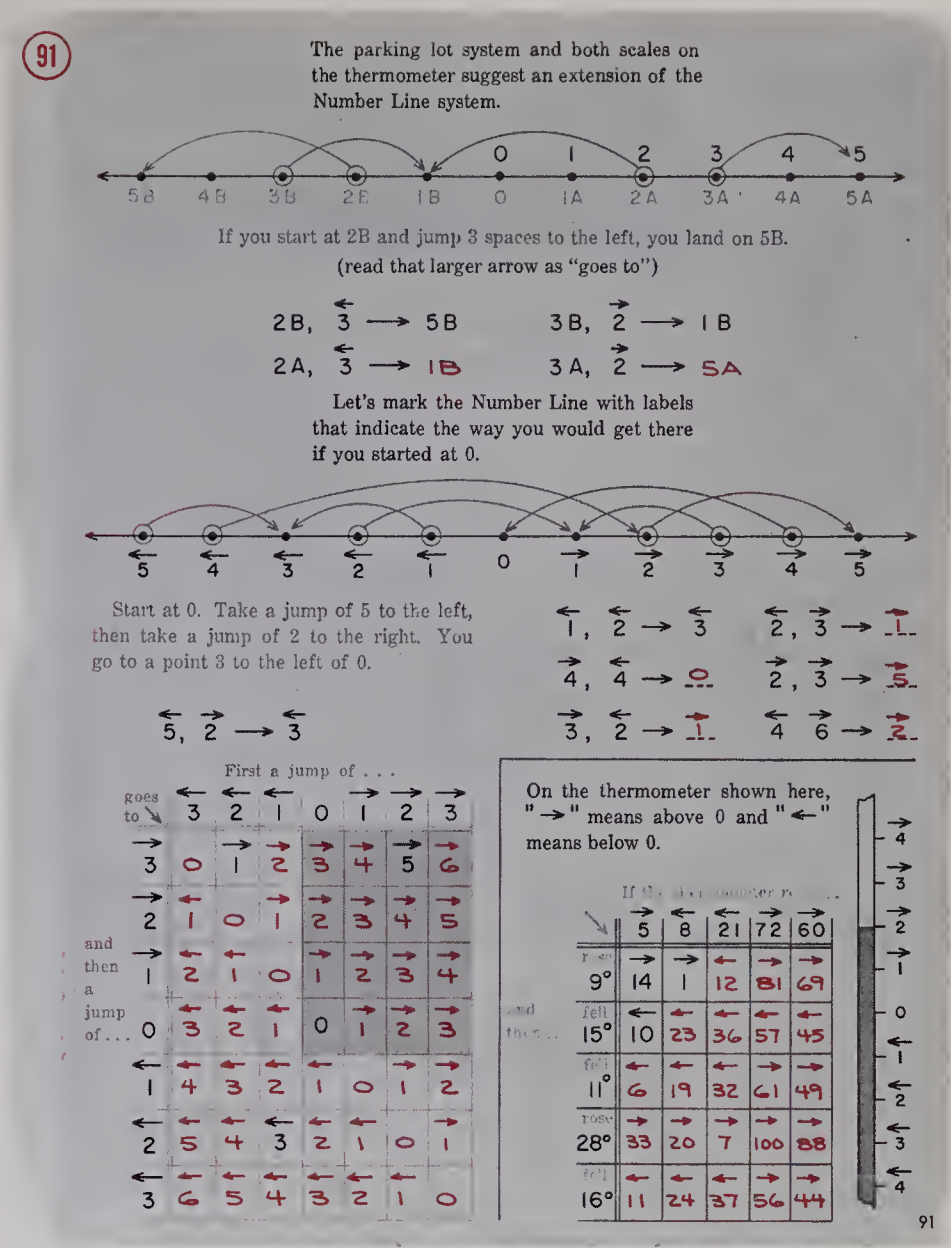
"Suppose we erase the labels on the line and use another system. Let's use a shorthand something like that of the chart of trips. Let's assign 0 to one point, and label the others to indicate what kind of trip you would take to get there from 0."



"How would you get from 2 ("left arrow 2") to 1 ("right arrow 1")?" Answer: Take a trip of 3 to the right.

"Suppose all the following trips start at 0. Each is a combination of 2 trips. 2 with a left-pointing arrow above it will mean two trips of 2 units to the left and 2 with a right-pointing arrow above it will mean a trip of two units to the right. The first trip starts from 0 and goes 3 units to the left; and from there a second trip goes 2 units to the right. Where does the trip end?" Answer: at 1 with a left-pointing arrow above it.

First trip	Second trip	Ends at
← 3 ,	→ 2	← 1
← 2 ,	→ 1	→ *3*
→ 2 ,	← 4	← *2*


91
The parking lot system and both scales on the thermometer suggest an extension of the Number Line system.
If you start at 2B and jump 3 spaces to the left, you land on 5B.
(read that larger arrow as "goes to")
2B, ← 3 → 5B
3B, → 2 → 1B
2A, ← 3 → 1B
3A, → 2 → 5A
Let's mark the Number Line with labels that indicate the way you would get there if you started at 0.
Start at 0. Take a jump of 5 to the left, then take a jump of 2 to the right. You go to a point 3 to the left of 0.
← 5, → 2 → ← 3
First a jump of . . .
goes to
and then a jump of . . .
On the thermometer shown here, "→" means above 0 and "←" means below 0.
9° 14 1 12 81 69
15° 10 23 36 57 45
11° 6 19 32 61 49
28° 33 20 7 100 88
16° 11 24 37 56 44
91

Without any further explanation, let children work through page 91 independently.

Notes to accompany *Work Text* page 92

Page 92 explores ways in which these "right arrow" and "left arrow" numbers reflect patterns in the world of events. Several interpretations are given:

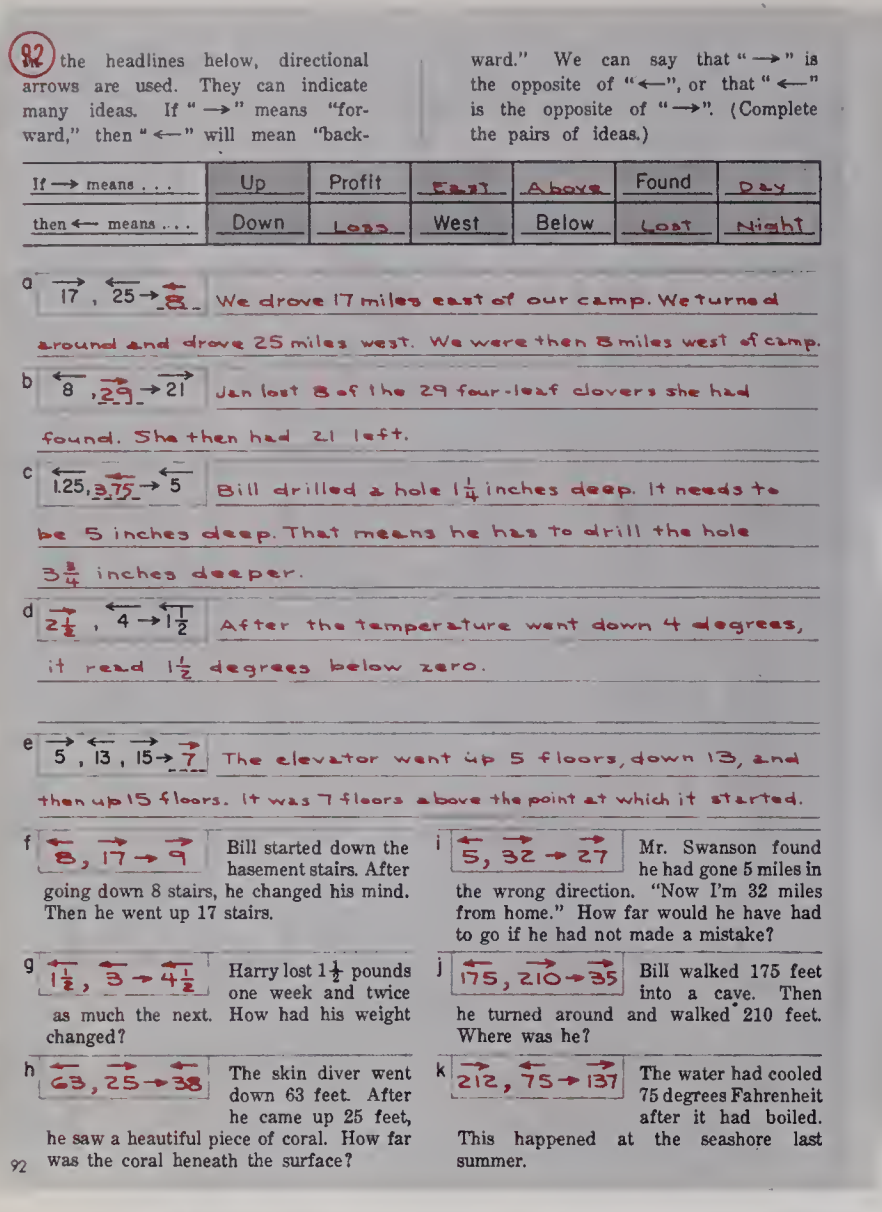
up and down
profit and loss
east and west
above and below
found and lost

Urge children to find others:

north and south
future and past
in and out
high and deep

In each case, several examples involving numbers should be found.

"I went up 13 stairs and down 3. I was 10 steps up from my starting point."



82 In the headlines below, directional arrows are used. They can indicate many ideas. If "→" means "forward," then "←" will mean "backward." We can say that "→" is the opposite of "←", or that "←" is the opposite of "→". (Complete the pairs of ideas.)

If → means . . .	Up	Profit	East	Above	Found	Day
then ← means . . .	Down	Loss	West	Below	Lost	Night

a 17 , 25 → 8 — We drove 17 miles east of our camp. We turned around and drove 25 miles west. We were then 8 miles west of camp.

b 8 , 29 → 21 — Jan lost 8 of the 29 four-leaf clovers she had found. She then had 21 left.

c 1.25, 3.75 → 5 — Bill drilled a hole $1\frac{1}{4}$ inches deep. It needs to be 5 inches deep. That means he has to drill the hole $3\frac{3}{4}$ inches deeper.

d $2\frac{1}{2}$, 4 → $1\frac{1}{2}$ — After the temperature went down 4 degrees, it read $1\frac{1}{2}$ degrees below zero.

e 5 , 13 , 15 → 7 — The elevator went up 5 floors, down 13, and then up 15 floors. It was 7 floors above the point at which it started.

f 8, 17 → 9 — Bill started down the basement stairs. After going down 8 stairs, he changed his mind. Then he went up 17 stairs.

g $1\frac{1}{2}$, 3 → $4\frac{1}{2}$ — Harry lost $1\frac{1}{2}$ pounds one week and twice as much the next. How had his weight changed?

h 63, 25 → 38 — The skin diver went down 63 feet. After he came up 25 feet, he saw a beautiful piece of coral. How far was the coral beneath the surface?

i 5, 32 → 27 — Mr. Swanson found he had gone 5 miles in the wrong direction. "Now I'm 32 miles from home." How far would he have had to go if he had not made a mistake?

j 175, 210 → 35 — Bill walked 175 feet into a cave. Then he turned around and walked 210 feet. Where was he?

k 212, 75 → 137 — The water had cooled 75 degrees Fahrenheit after it had boiled. This happened at the seashore last summer.

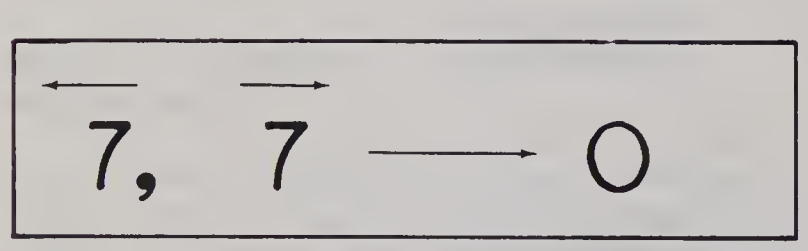
92

"My birthday was a week ago, and my sister's is the day after tomorrow. Our birthdays are 9 days apart."

"The point of a stake is 10 inches in the ground. The stake is 2 feet long. The top of the stake is 14 inches above the ground."

"The temperature fell from 22° above zero to 19° above zero. This is a fall of 3°."

Next, use a headline story technique to strengthen understanding of ways in which these numbers help express familiar patterns:

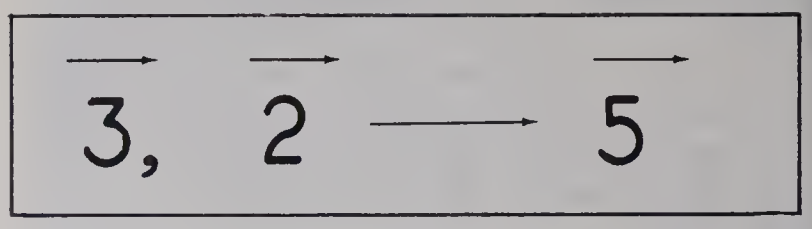


"I walked 7 blocks west to my grandmother's, and then walked 7 blocks east—which brought me back home."

"Bill dove down 7 feet in the water and came back up to the surface."

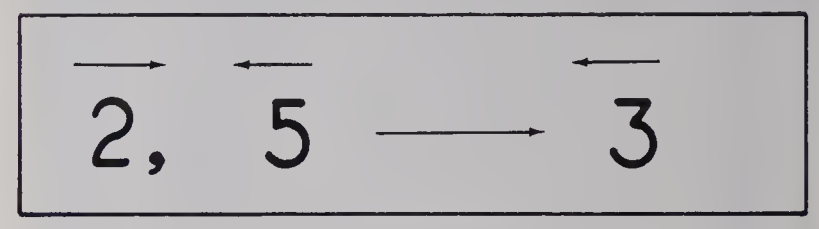
"I went into the cave 7 feet and back out."

"The temperature fell 7° and then rose 7°."



"A trip of 3 miles to the east and then 2 miles more to the east is the same as a 5 mile trip to the east."

"I piled 2 feet of dirt on top of a pile that was already 3 feet high. It was then 5 feet high."



"Harry gained 2 pounds and lost 5 pounds. He weighs 3 pounds less than before."

"Our team gained 2 yards on one play and lost 5 yards on the next. That's a net loss of 3 yards."

After considerable class participation in creating such examples and discussing them, most of the children can work successfully by themselves through page 92.

Notes to accompany *Work Text* pages 93-94

First, we shift our symbolism and language to the accepted convention of "+ 3" read "positive three" and "− 4" read "negative four" . . . but we point out that there is no change at all in what these symbols and expressions imply.

All examples we have been exploring have had a certain common aspect—involving two trips, two changes on the thermometer. The idea running through them all we call *addition*. With that idea in mind, we write:

$(+1) + (+1) = (+2)$
"positive one plus positive one equals positive two"
$(-1) + (+3) = (+2)$
"negative one plus positive three equals positive two"

"Directed" number is one of the names that has been given to these numbers that can be used to label points on the Number Line on both sides of 0. Where do you think that name came from?

Others call them "signed" numbers. Why might that be an appropriate name?

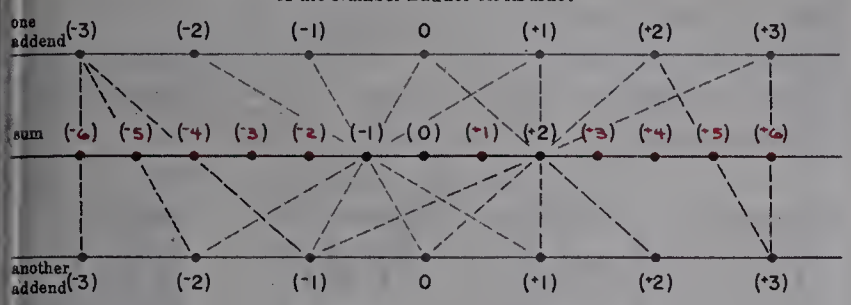
Most mathematicians do not use arrows such as we have used ... $\overleftarrow{3}$ or $\overleftarrow{4}$: They prefer to indicate a direction from 0 in the following way:

(+3) . . . read "positive three," and
(-4) . . . read "negative four."

We will begin using these preferred signs.

What do you suppose we can mean by ADDITION of Directed Numbers?

Here is a picture using the idea of the Number Ladder on its side:



Can you finish labelling the middle line?

(+1) + (+1) = (+2)	(+1) = (-1) + (+2)	(+3) + (-2) = (-2) + (+3)
(0) + (+2) = (+2)	(-2) = (-3) + (+1)	(-2) + (-3) + (+4) = (-1)
(+2) + (0) = (+2)	(-3) + (+3) = (0)	(+4) + (-2) + (-3) = (-1)
(+3) + (-1) = (+2)	(0) = (-1) + (+1)	(+2) + (-7) = (-3) + (-2)
(-2) + (+1) = (-1)	(+3) + (+2) = (+5)	(+9) + (-5) = (-5) + (+9)
(-1) + (0) = (-1)	(-2) + (+4) = (+2)	(-4) + (-4) + (-4) = (-12)
(0) + (-1) = (-1)	(-5) = (-1) + (-4)	(+5) + (+5) + (+5) = (+15)
(+1) + (-2) = (-1)	(-15) = (-7) + (-8)	(-17) + (-8) + (+17) = (-8)
(-3) + (-1) = (-4)	(-6) + (+15) = (+9)	(-25) + (+9) + (+16) = (0)
(-3) + (-2) = (-5)	(+6) = (+15) + (-9)	(+3) = (-7) + (+11) + (-1)
(-3) + (-3) = (-6)	(-10) + (+3) = (-7)	(-2) + (+10) + (-12) = (-4)

93

We use parentheses so that the signs for "positive" and "negative" do not become confused with the signs for the operations of addition and subtraction.

Work Text page 93 presents a number ladder. Consider a pair of addends, such as (+ 3) and (− 1). Locate one addend on the top number line, and the other on the lower number line. Connect these points with a straight line. Mark the point at which their connection crosses the center line, and label the point to indicate the sum: (+ 3) + (− 1) = (+ 2). So, in this case, label the point (+ 2). Keep following this procedure until all the points are labeled on the center line.

The pattern of labeling on the center line turns out to be most familiar. And we are, perhaps, a little surprised to find that the point marked (+ 2) on this center line will be the point at which the line connecting any pair of addends whose sum is (+ 2) crosses that line. So, we have constructed a kind of adding machine for these "directed numbers." Using this device, most of the examples on the bottom of page 93 can be illustrated by a line on the number ladder.

A different sketch to illustrate the idea of addition of directed numbers is developed on page 94.

The idea of addends indicating "trips," starting from 0 and taken one after the other, is utilized. The first trip is represented by a wavy line; the second by a straight line of the same color. By inspection, we find, for the "sum," the label of the point reached by taking these trips. In each example, the first trip is understood to be indicated by the addend mentioned first. This gives a certain importance to order, so we consider it.

Is it true that (− 3) + (+ 5) = (+ 5) + (− 3)? If we cannot be convinced that this is true, we will be in trouble—we must learn a lot of new facts of arithmetic. Fortunately, our study is quite persuasive that order is of no consequence. So the pattern suggested for whole numbers by the expression that, for each a and each b, a + b = b + a, also holds for directed numbers as well. (Mathematicians refer to this as the "commutative law" of addition.)

You may want to explore the meaning of expressions such as:

(+ 2) − (+ 3) = ____
(+ 2) − (− 3) = ____
(− 2) − (+ 3) = ____
(− 2) − (− 3) = ____

Consider, at this point, noting that in each instance we are given a sum and an addend—and, an addend is missing. We can rewrite each expression in terms of addition:

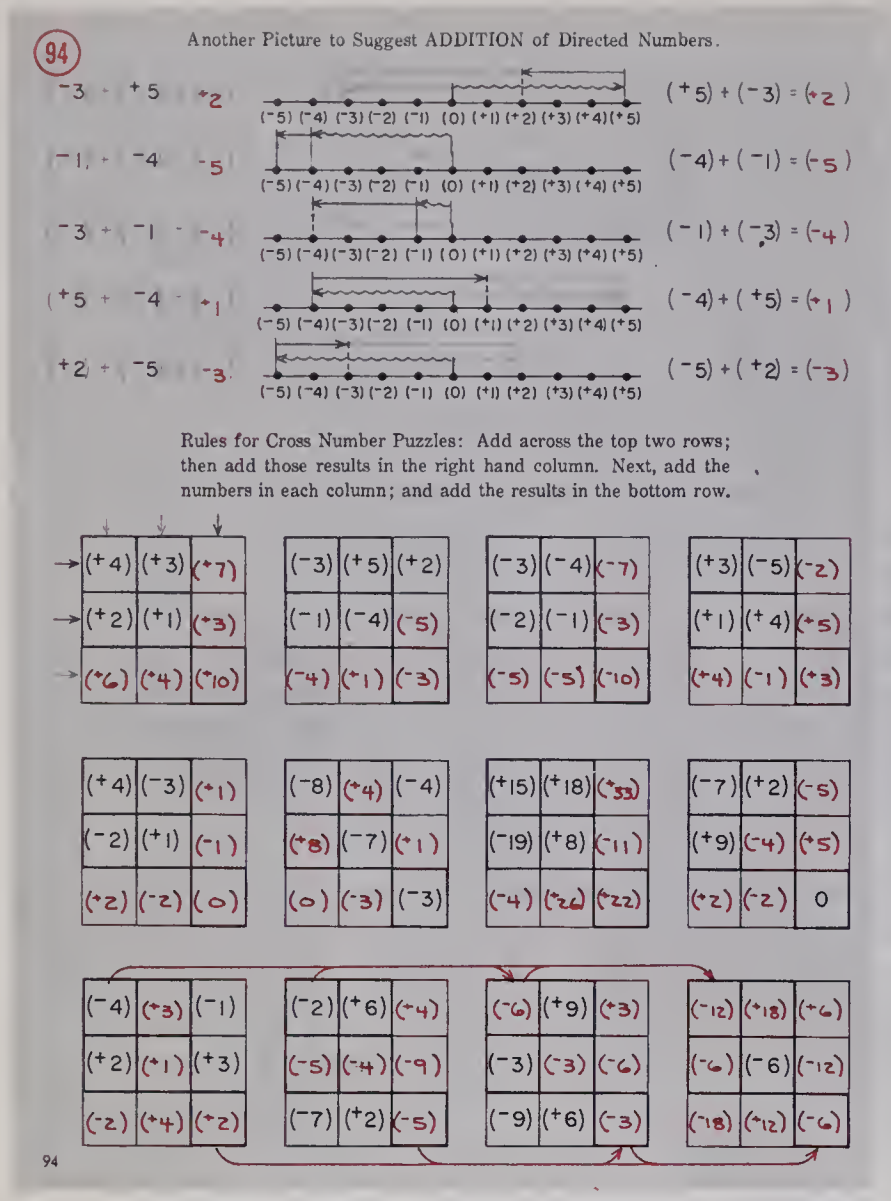
(+ 2) − (+ 3) = ____ can be written
(+2) + ____ = (+ 3)
(+ 2) − (− 3) = ____ can be written
(+2) + ____ = (− 3)
(− 2) − (+ 3) = ____ can be written
(−2) + ____ = (+ 3)
(− 2) − (− 3) = ____ can be written
(−2) + ____ = (− 3)

In this form, we can find the missing addend without considering a special meaning for subtraction.

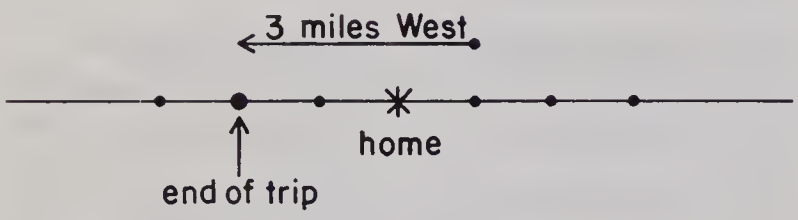
This approach also gives us a basis for giving a clear indication of patterns in life to fit such headlines as:

(-2) − (-3) = (+1)

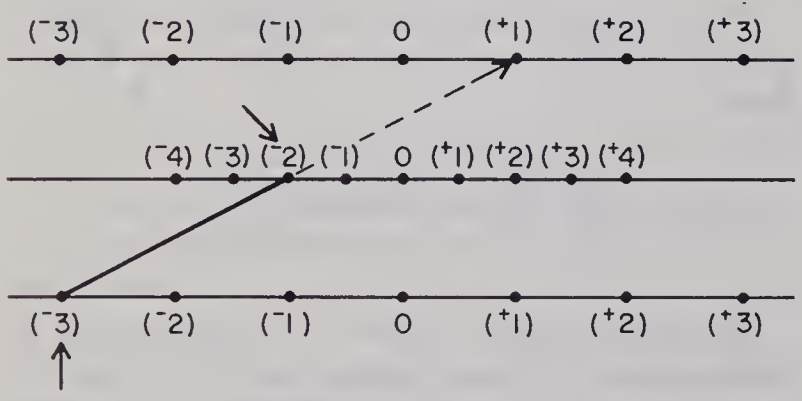
In terms of "trips," the (− 2) certainly refers to the point at which the combination of trips ended—it is the "sum" of two trips. Similarly, (− 3) refers to one of the trips. So, we interpret that headline as: "I ended 2 miles west of home. One of the trips I


94
Another Picture to Suggest ADDITION of Directed Numbers.
Rules for Cross Number Puzzles: Add across the top two rows; then add those results in the right hand column. Next, add the numbers in each column; and add the results in the bottom row.
94

took was a trip 3 miles west. What was my first trip that started at home?" Answer: a trip of 1 mile east.



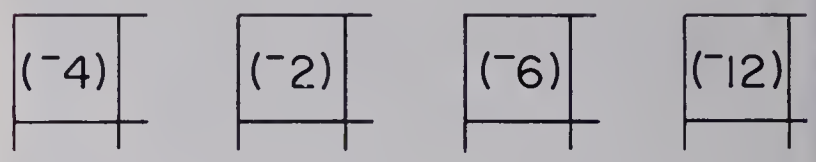
Another way to look at the matter is to consider the number ladder on page 93. Find (-2) on the center line—since it is a "sum." Locate (-3) on the top or bottom line. Connect these points and extend the line to "find the other addend."



Again, we are led to argue that $(-2) - (-3) = (+1)$.

The lower section on *Work Text* page 94 is a simple use of the basic Cross Number Puzzle form—and the general rule of addition. Most of the individual examples of addition involved can be sketched on the number line above or can be illustrated on the number ladder.

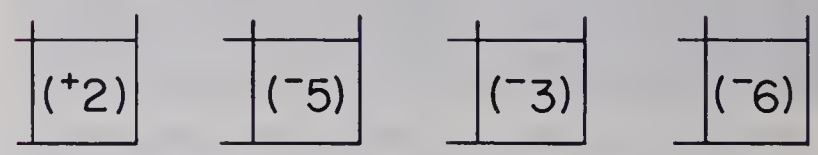
In the bottom row of four puzzles, an interesting pattern can be uncovered. Consider any group of four corresponding cells, such as those in the upper left-hand corner:



$$(^-4) + (^-2) = (^-6)$$

$$(^-4) + (^-2) + (^-6) = (^-12)$$

All nine sets of corresponding cells follows this same pattern—including the check box in the lower right-hand corner:



$$(^+2) + (^-5) = (^-3)$$

$$(^+2) + (^-5) + (^-3) = (^-6)$$

This pattern is for the faster workers to explore.

Notes to accompany *Work Text* page 95

On page 95, a magic square of "directed numbers" is given in part in the upper left hand corner of the page. The magic number is $(+15)$. Knowing this, the square can be filled in.

Now we turn to the blank square in the upper right-hand corner. Directions are that the number to be written in each cell should be found by "adding (-5)" to the number written in the corresponding cell of the other square:

$$(+6) + (-5) = (+1)$$
$$(+7) + (-5) = (+2)$$
$$(+2) + (-5) = (-3)$$

In a MAGIC SQUARE, the sum of the numbers in each row, column and diagonal are the same.

(+15)	(+6)	(+7)	(+2)
(+15)	(+1)	(+5)	(+9)
(+15)	(+8)	(+3)	(+4)
(+15)	(+15)	(+15)	(+15)

To find the number in each cell of the Magic Square on the left,

ADD (−5) and

write the result in the respective cell on the right.

(+1)	(+2)	(−3)	(0)
(−4)	(0)	(+4)	(0)
(+3)	(−2)	(−1)	(0)
(0)	(0)	(0)	(0)

Example: (+6) + (−5) = (+1)

You may be surprised at the result above.

Add each pair of similarly located numbers in the Magic Squares above.

Examples:

(+6) + (+1) = (+7) (+7) + (+2) = (+9)

(+1) + (−4) = (−3) (+5) + (0) = (+5)

(+8) + (+3) = (+11) (+3) + (−2) = (+1)

Is the result a Magic Square?

(+15)	(+15)	(+15)	(+15)
(+7)	(+9)	(−1)	(+15)
(−3)	(+5)	(+13)	(+15)
(+11)	(+1)	(+3)	(+15)
(+15)	(+15)	(+15)	(+15)

	A	B	C
X	(+11)	(+2)	(−3)
Y	(−4)	(0)	(+4)
Z	(+3)	(−2)	(−1)

Here is A Game by Sue Marson that the whole Arithmetic Class can play.

Team I has these cards: X Y Z

Team II has these cards: A B C

Team I wins

Team II wins

(−10) (−9) (−8) (−7) (−6) (−5) (−4) (−3) (−2) (−1) (0) (+1) (+2) (+3) (+4) (+5) (+6) (+7) (+8) (+9) (+10)

Each of two teams has a set of 3 cards — X, Y and Z or A, B, C. Each team selects one and places it on the table face down. Both are turned up — such as X by Team I and C by the other team.

Look at the Magic Square above. What number is in row X and column C? The answer (−3). Score is kept on the Number Line above as shown.

Teams select another card (or the same one) — such as C and Y which indicates a score of (+4). This is noted on the Number Line scoreboard.

If the next pair of cards were B and X, that moves the score (+2) — or, 2 to the right in this case. If AZ is next, the score moves 3 more to the right. If CY were next — (+4) — that calls for a jump to the right, landing at (+10), and Team II would win.

96 Complete the chart in each example. Then select one of the facts in the summary as a Headline for a story.

In the first example you might select the fact that 20% x 50 = 10 or 30% of 500 = 150.

X	5	10	50	250
10%	.5	1	5	25
20%	1	2	10	50
3%	.15	.30	1.5	7.5

There were 50 people at the meeting. 20% of them took part in the discussion. Ten people took part.

Bill sold 17 stamps at 6¢ each, and the same number at 9¢ each. He received $2.55 from the sale.

X	10	17	27	44
6	60	102	162	264
9	90	153	243	396
15	150	255	405	660

+	.7	1.15	9.82
.7	1.4	1.85	10.52
.69	1.39	1.84	10.51
3.58	4.28	4.73	13.40
10.25	10.95	11.40	20.07

Mrs. Powell bought two dresses. One cost $9.82 and together they cost $20.07. She spent $10.25 for the other dress.

After Henry had walked a quarter of a mile, he had a half-mile left to go. His whole trip would be three-quarters of a mile.

−	$\frac{3}{4}$	$1\frac{3}{8}$	$\frac{9}{16}$	2
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{7}{8}$	$\frac{1}{16}$	$1\frac{1}{2}$
$\frac{1}{4}$	$\frac{1}{2}$	$1\frac{1}{8}$	$\frac{5}{16}$	$1\frac{3}{4}$
$\frac{1}{8}$	$\frac{5}{8}$	$1\frac{1}{4}$	$\frac{7}{16}$	$1\frac{7}{8}$

Ways	1	2	3	4	5	6	7	8
Mary	1	2	3	4	5	6	7	
Jane	0	1	2	3	4	5	6	
John	6	5	4	3	2	1	0	
Bill	6	5	4	3	2	1	0	
Total	13	13	13	13	13	13	13	

You Write the Headline.

One day, the Bowman children counted the books they brought home from the library. There were 13 in all.

"I brought 1 more than Jane," Mary said.

"John brought home the same number that I did," remarked Bill.

There are only 7 ways in which the numbers of books can be distributed so that the above statements are true.

Thus we have the top row of the new square. We can add these numbers to find the new magic number:

$$(+1) + (+2) + (-3) = (0)$$

Quite a surprise. The square can be completed in similar fashion. Next, consider the third magic square grid. Each cell in this square is to contain the sum of the two numbers written in the corresponding cells of the squares above. The examples written out on the page lead to the results for the left and center columns in the new square.

The magic number turns out to be (+ 15), as in the original square. "How can you explain this?" This question may stimulate some to independent study of magic squares involving "directed numbers."

The lower part of page 95 is devoted to a game by Sue Marson—her version of an old game. This is a game the whole class can learn to play.

Put the grids on the board. Prepare the six cards as shown—X, Y, Z for one team and A, B, C for the other. The number line shown is the scoreboard for the game. Rules are given on the page.

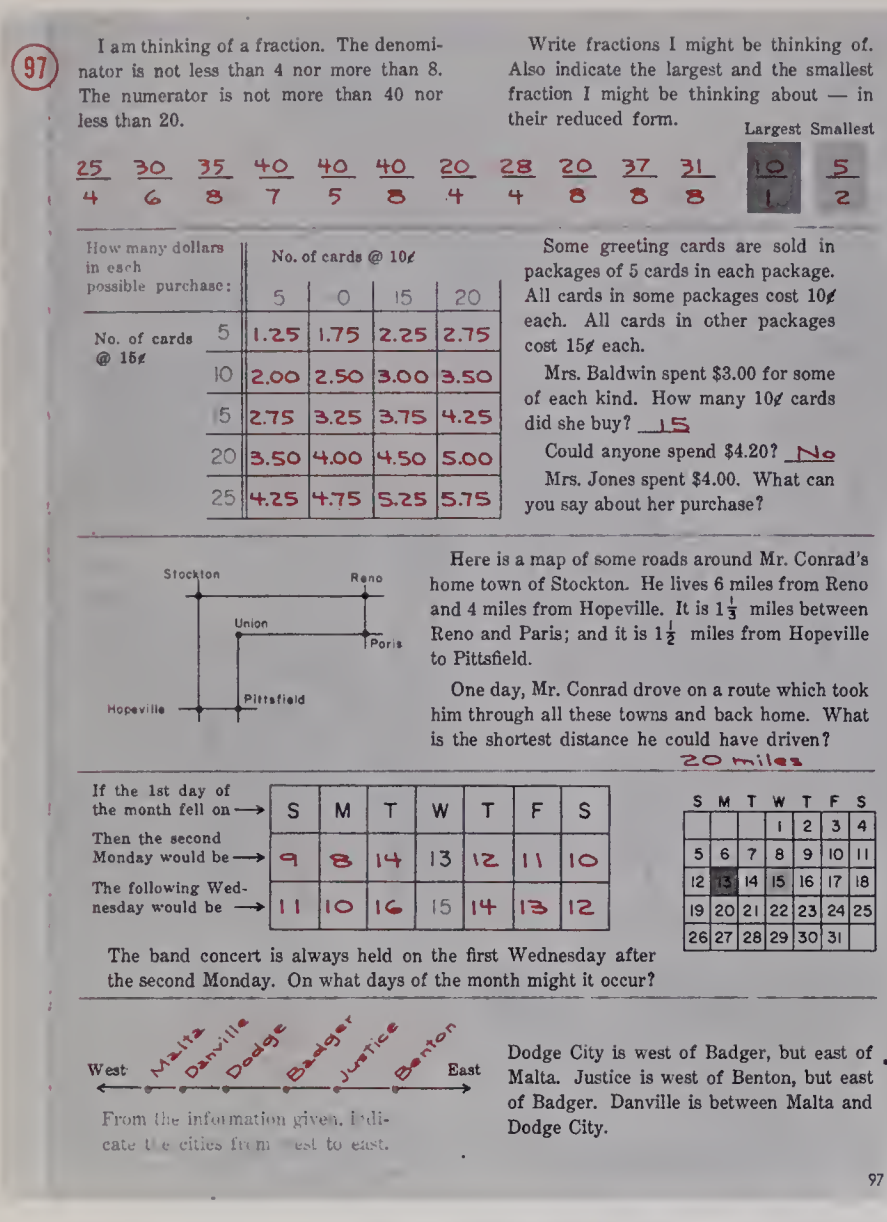
Notes to accompany *Work Text* pages 96-97

These two pages explore an extension of the headline story technique. In the examples given, charts or maps or pages from a calendar or a series of fractions reveal a certain pattern—as headlines. What stories might fit the headlines given? Or, in what way are the headlines appropriate for the stories given?

In some cases, the story is a puzzle that would be most difficult to solve without the headline. The first example on page 96 has an incomplete headline—a chart with only one cell filled in. The first job, as usual, is to complete the headline. "Multiplication" is indicated in the corner of the chart.

Now, we need a story to fit the headline. Since the blue cell is singled out, we begin with a story to fit that part of the chart. "There were 50 people at the meeting. 20% took part in the discussion. So, 10 people must have taken part." How would that story change if we directed our attention to other parts of the chart?

"There were 250 people at the meeting and 10% took part. Twenty-five people participated actively."


97
I am thinking of a fraction. The denominator is not less than 4 nor more than 8. The numerator is not more than 40 nor less than 20.
Write fractions I might be thinking of. Also indicate the largest and the smallest fraction I might be thinking about — in their reduced form.
Largest Smallest
25/4 30/6 35/8 40/7 40/5 40/8 20/4 28/4 20/8 37/8 31/8
10/1 5/2
How many dollars in each possible purchase:
No. of cards @ 10¢
No. of cards @ 15¢
5 0 15 20
5 1.25 1.75 2.25 2.75
10 2.00 2.50 3.00 3.50
15 2.75 3.25 3.75 4.25
20 3.50 4.00 4.50 5.00
25 4.25 4.75 5.25 5.75
Some greeting cards are sold in packages of 5 cards in each package. All cards in some packages cost 10¢ each. All cards in other packages cost 15¢ each.
Mrs. Baldwin spent $3.00 for some of each kind. How many 10¢ cards did she buy? 15
Could anyone spend $4.20? No
Mrs. Jones spent $4.00. What can you say about her purchase?
Stockton
Reno
Union
Paris
Hopeville
Pittsfield
Here is a map of some roads around Mr. Conrad's home town of Stockton. He lives 6 miles from Reno and 4 miles from Hopeville. It is 1 1/3 miles between Reno and Paris; and it is 1 1/2 miles from Hopeville to Pittsfield.
One day, Mr. Conrad drove on a route which took him through all these towns and back home. What is the shortest distance he could have driven?
20 miles
If the 1st day of the month fell on →
Then the second Monday would be →
The following Wednesday would be →
S M T W T F S
9 8 14 13 12 11 10
11 10 16 15 14 13 12
S M T W T F S
1 2 3 4
5 6 7 8 9 10 11
12 13 14 15 16 17 18
19 20 21 22 23 24 25
26 27 28 29 30 31
The band concert is always held on the first Wednesday after the second Monday. On what days of the month might it occur?
West
Malta
Danville
Dodge
Badger
Justice
Benton
East
From the information given, indicate the cities from west to east.
Dodge City is west of Badger, but east of Malta. Justice is west of Benton, but east of Badger. Danville is between Malta and Dodge City.
97

Are there some parts of the chart that would not lend themselves as easily to this situation? Yes. It would be hard to use $10\% \times 5 = .5$ as a headline without changing the story, since we would find ourselves referring to ".5 of a person" which is meaningless in this situation.

We might use that same item on the chart as a title for, "I spent 10% of $5.00 savings, and now have $4.50 left."

Thus, each headline-story example can be used for many versions of the same story, as well as for many different stories. The first four examples on page 96 and the second on page 97 can easily become the subject of lively class discussions and story telling.

The last example on page 96 is more of a puzzle. The story is given. The headline is in the form of a chart to record the possibilities the story outlines. Without this assistance, the question at the end of the story might remain unsolved.

Mary said she bought "one more than Jane." We don't know how many Jane brought home, but 0 is the fewest she could have brought. So we list all the possibilities—0 and 1, 1 and 2, 2 and 3 . . . up to 6 and 7. Here we stop, because the total all brought home was 13 and $6 + 7 = 13$. In each case, we can also note the number John and Bill brought home. If Jane brought home none, and Mary brought home "one more," then they brought a total of 1 because $0 + 1 = 1$. Then Bill and John must have brought home 12 between them, because $13 - 1 = 12$. And, since we know the boys each brought home the same number, it must have been 6 apiece, because $12 \div 2 = 6$.

Finally, we find that there are 7 ways in which the 13 books might have been brought home—and all given conditions met.

A somewhat similar pattern must be uncovered in the fourth example on Page 97. The common thread is that all possible combinations must be explored.

The first of the month can fall on any day of the week. So, we find the days of the month that might be the "second Monday." Of course, the earliest date would occur if the month started on Monday—and the second Monday would be the 8th. The latest date for the second Monday happens when the month starts on Tuesday—and the first Monday would be the 7th, making the second Monday the 14th.

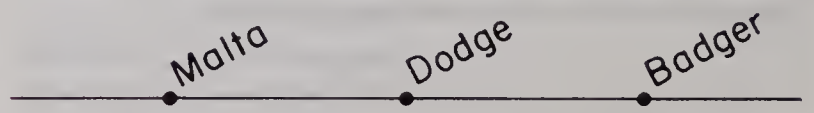
So the second Monday may be as early as the 8th or as late as the 14th. Since Wednesday is two days later, it can be as early as the 10th or as late as the 16th—and can fall on any date in between. Thus, the answer to our problem is, "the 10th, 11th, 12th, 13th, 14th, 15th or 16th." There are seven possibilities.

The final example on page 97 uses few words, but after you read them there is utter confusion. A sketch is almost indispensable for more people to unravel this short series of statements. Start with an east-west line:

Now arrange the information, bit by bit:

"Dodge City is west of Badger, but east of Malta."

a—Dodge is west of Badger
b—Dodge is east of Malta



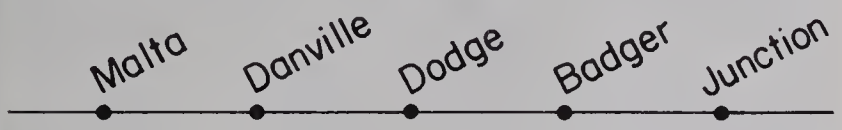
We get that information down.

"Junction City is west of Benton, but east of Badger."

c—Junction is west of Benton
d—Junction is east of Badger

We can get the idea of *d* down on our line, but we haven't heard of Benton before.

"Danville is between Malta and Dodge."



Now we return to *c*—"Junction is west of Benton." That is the same as saying that Benton is east of Junction—and that places it sufficiently:

"Can they be arranged in any other order and still follow the pattern set forth in those three short sentences? . . . A question for independent study.

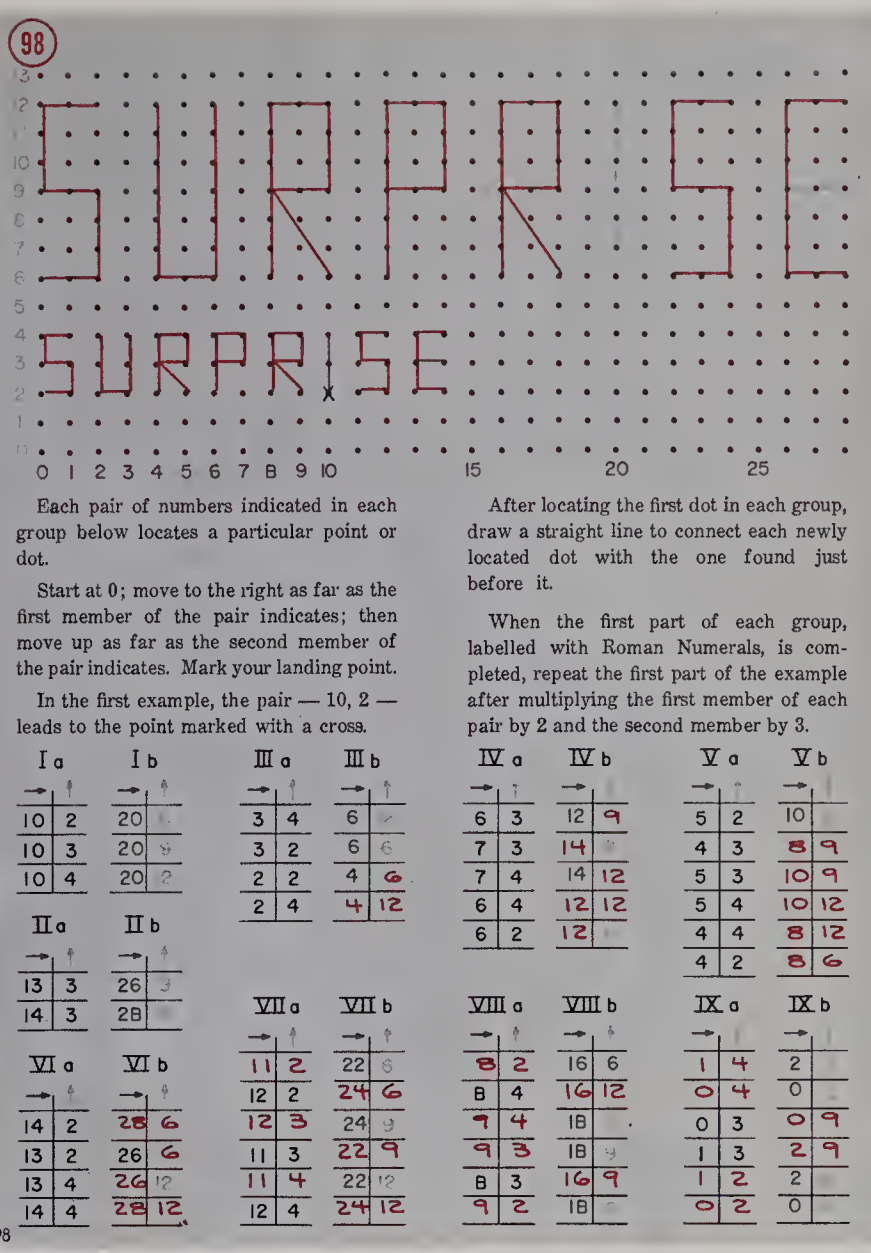
Notes to accompany *Work Text* pages 98-99

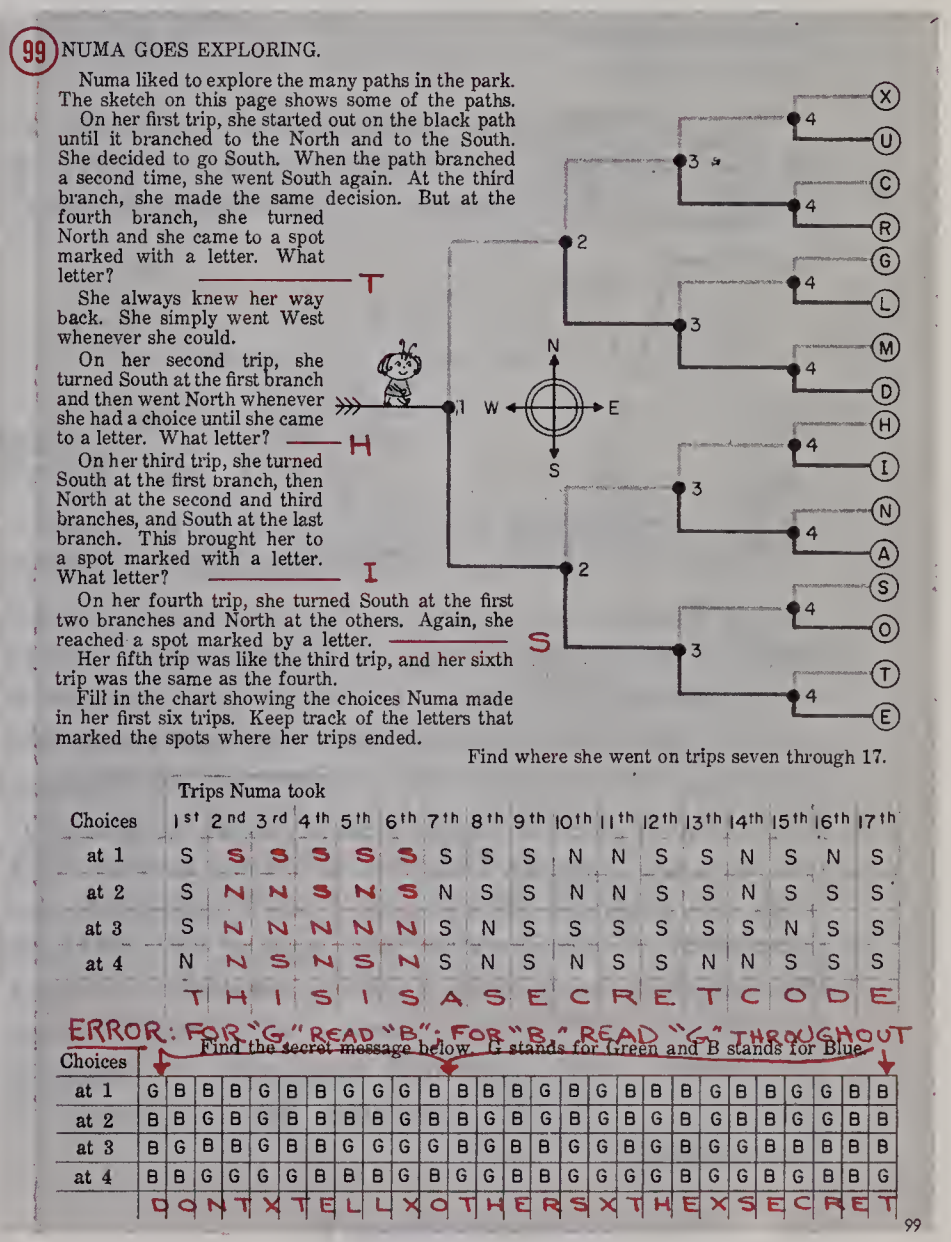
Pages 98 and 99 are for independent study. Each has some of the elements of puzzle in it and a bit of surprise. Let each child discover the secrets for himself.

Notes to accompany *Work Text* page 100

Pages 100 and 101 are intended to arouse the students' interest by introducing them to some surprising facts about triangles and four-sided figures (quadrilaterals). Although these two pages do not attempt to explain the *why* of these surprises, page 102 does explain how to measure angles in degrees; and pages 103 through 106 do introduce the students to a few facts about angles that will help them to understand the surprises on these two pages.

We are, of course, not trying to deal with geometry in the spirit of Euclid. Instead, we are dealing with a few simple experiments which involve tearing off the corners of paper, and pasting them together again. These experiments acquaint the student with



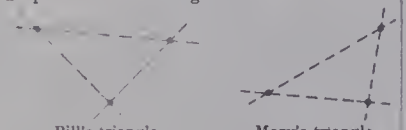

99
NUMA GOES EXPLORING.
Numa liked to explore the many paths in the park. The sketch on this page shows some of the paths.
On her first trip, she started out on the black path until it branched to the North and to the South. She decided to go South. When the path branched a second time, she went South again. At the third branch, she made the same decision. But at the fourth branch, she turned North and she came to a spot marked with a letter. What letter? T
She always knew her way back. She simply went West whenever she could.
On her second trip, she turned South at the first branch and then went North whenever she had a choice until she came to a letter. What letter? H
On her third trip, she turned South at the first branch, then North at the second and third branches, and South at the last branch. This brought her to a spot marked with a letter. What letter? I
On her fourth trip, she turned South at the first two branches and North at the others. Again, she reached a spot marked by a letter. S
Her fifth trip was like the third trip, and her sixth trip was the same as the fourth.
Fill in the chart showing the choices Numa made in her first six trips. Keep track of the letters that marked the spots where her trips ended.
Find where she went on trips seven through 17.
N
W
E
S
1
2
3
4
X U C R G L M D H I N A S O T E
Trips Numa took
Choices
1st 2nd 3rd 4th 5th 6th 7th 8th 9th 10th 11th 12th 13th 14th 15th 16th 17th
at 1 S S S S S S S S S N N S S N S N S
at 2 S N N S N S N S S N N S S N S S S
at 3 S N N N N N S N S S S S S S N S S
at 4 N N S N S N S N S N S S N N S S S
T H I S I S A S E C R E T C O D E
ERROR: FOR "G" READ "B"; FOR "B," READ "G" THROUGHOUT
Find the secret message below. G stands for Green and B stands for Blue.
Choices
at 1 G B B B G B B G G G B B B B G B G B B B G B B G G B B
at 2 B B G B G B B B B G B B G B G B G B G B G B B G G B B
at 3 B G B B G B B G G G G B G B B G G B G B G G B B B B B
at 4 B B G G G G B B B G B G G B B G G G G B G G B G B B G
D O N T X T E L L X O T H E R S X T H E X S E C R E T

100

Surprises

"Here's a trick," Mary explained to Bill, "that has an interesting surprise to it. Mark three points on your piece of paper and draw a line through each pair of points. Then cut along those lines and a piece will drop out.

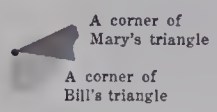
"Sure," Bill said, "and the pieces that drop out will be triangles."



Bill's triangle Mary's triangle

"Now color each corner, and then tear the triangle into three parts, a corner to each part.

"Next," Mary continued, "I will paste one of my pieces with the marked point touching this dot I am drawing. Then you paste one of your pieces with its point on the dot — and right next to mine.



"Then I will paste another corner of my triangle next to yours; you do the same; and I will paste my last corner down. And you paste your last corner down."

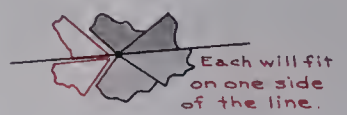
"It fits!" Bill exclaimed. "Will wonders never cease!"

"Well," Mary said with a twinkle in her eye, "here is another surprise for you. Start out just as we did before. Draw any shape triangle you like, and so will I. Mark its corners, and tear it so each piece contains one corner.

"This time, I'll mark a point on a line and paste one of my corners on one side of the line and you do the same on the other side."

Here is the result:

"Then I'll paste a second corner next to my first and you do the same on your side."

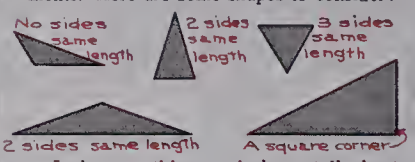


"Finally, I'll paste my last corner down and you do the same," Mary explained.

"Amazing!" Bill said. "Let's try it again, but this time I'm going to change the shape of my triangle."

What was the surprise?

Try it yourself. Cut out two triangles of any shape and repeat both of the experiments. Here are some shapes to consider:



Is there anything particular to talk about in each of these triangles?

101

Bill's Surprises

The next day, Bill came to class with a surprise of his own.

"Mary," he said, "I worked out a trick something like yours. Mark any four points and connect them so you have a four-sided figure. Like this

"Now, cut along the line that connects the points. Next, tear each shape into four pieces so that each contains one of the corners — one of the original four points.

"Then, arrange each group of four pieces so that the corners touch at the points, and so that their sides just touch.

"Surprised?" Bill asked.

"I surely am surprised," Mary replied. "Would it work for squares, rectangles, and other four-sided shapes?"

"Try it!"

"The other fitting trick is this: Cut the corners of a rectangular piece of paper so that the corner cut-offs are different shapes and sizes.

"Each triangle corner cut-off has a square corner. Tear each triangle in three pieces so that one of the corners is in each piece. Try to fit the other corners of each into the square corner.

"Try this experiment in 'fitting' with each of the triangular pieces that has a square corner."

"It's a 'fitting' surprise," Mary agreed.

Tom has been watching and trying some examples of his own.

"I think all the results would be true," he said, "even if you turned all the pieces over or just some of them."

"That's hard to believe," Mary said.

"Try it, then," said Tom.

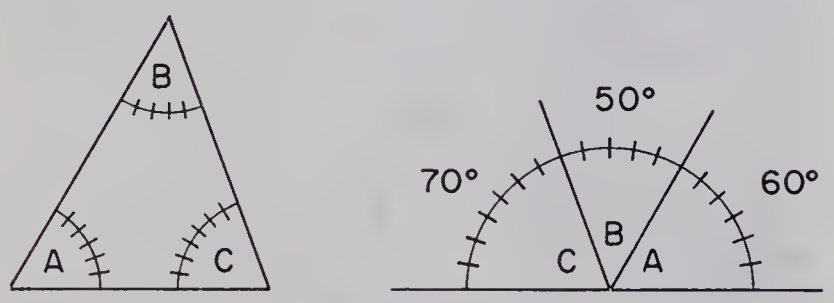
Make a study of your own of these fitting tricks. Try each trick with many different shapes. Turn some of the pieces over.

As you try each experiment, paste down your results so they become a record of these examples of 'fitting.'

Perhaps you can invent a 'fitting trick' of your own.

results that will help him later to appreciate the shapes of objects.

The chief idea of the experiments is pictured at the bottom of page 103 where the three corners of a triangle are fitted together to fill one side of a line:



This fact, which applies to any triangle, can be stated in terms of numbers. If we take a protractor which has 180 degrees, we find that the corners of a triangle (as shown on page 103) will just cover the protractor. So, together the corners fill 180 degrees. In the figure above, each division mark on the protractor represents 10 degrees. If corner A were torn from the triangle shown above and pasted on the protractor, it would cover 60 degrees. In the same way, if corners B and C were similarly cut and pasted, B would cover 50 degrees, and C would cover 70 degrees. The combined angles of the corners ($60 + 50 + 70 = 180$) just cover the 180 degrees of the protractor.

Mary's second trick on page 100 demonstrates the same fact. She too fills one side of a line with the corners of a triangle. Bill fills the other side of the line in the same way, thereby repeating the same experiment.

Mary's second trick explains her first trick. The angles of Mary's triangle fill 180 degrees; the angles of Bill's triangle also fill 180 degrees. The angles of both triangles fill 360 degrees, which corresponds to a complete circle. By mixing Bill's angles with her own, Mary proves that the total space taken has nothing to do with the order in which the pieces are assembled.

These points need not be emphasized in class. For the most part, the discussion in the student's text will probably be sufficiently clear. Certainly the material on pages 103-106 will be helpful. After reading these pages, students will probably grasp the connection that exists between the work on pages 103-106 and the work on pages 100 and 101. It is better to let these ideas occur gradually to the students than to offer a full explanation immediately.

Notes to accompany *Work Text* page 101

The first column on page 101 is related to facts that will be discovered on *Work Text* page 106.

The second column deals with triangles that have one angle of 90 degrees. Since the three angles of a triangle must contain 180 degrees, any 90-degree angle must take as much space as the other two angles put together. This is proved by showing how the "square angle" (right angle) is big enough for the other two angles to fit into it exactly. Do not feel obligated to explain why the tricks on this page work as they do.

Notes to accompany *Work Text* page 102

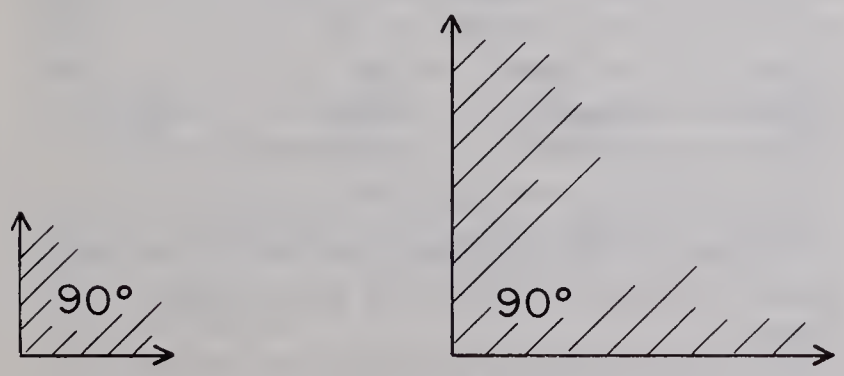
Page 102 deals with an angle and its measurement. We begin with the idea of "amount of turning." Have students imagine how much the hour hand of a clock turns between 12 o'clock and 1 o'clock. Then ask if they can think of other turns the hour hand makes that would be equal to this. (4 o'clock to 5 o'clock). What examples can they think of that would mean the hour hand had turned twice as far (12 o'clock to 2 o'clock)?

A circular clock face is divided into 60 equal parts. A protractor, also divided into equal parts, is designed on the basis that a circle contains 360 degrees. This means that every minute on a circular clock face corresponds to 6 degrees on a protractor. In other words, between 12 o'clock and 1 o'clock the hour hand turns through 30 degrees; and between 12 o'clock and 3 o'clock, the hour hand turns through 90 degrees.

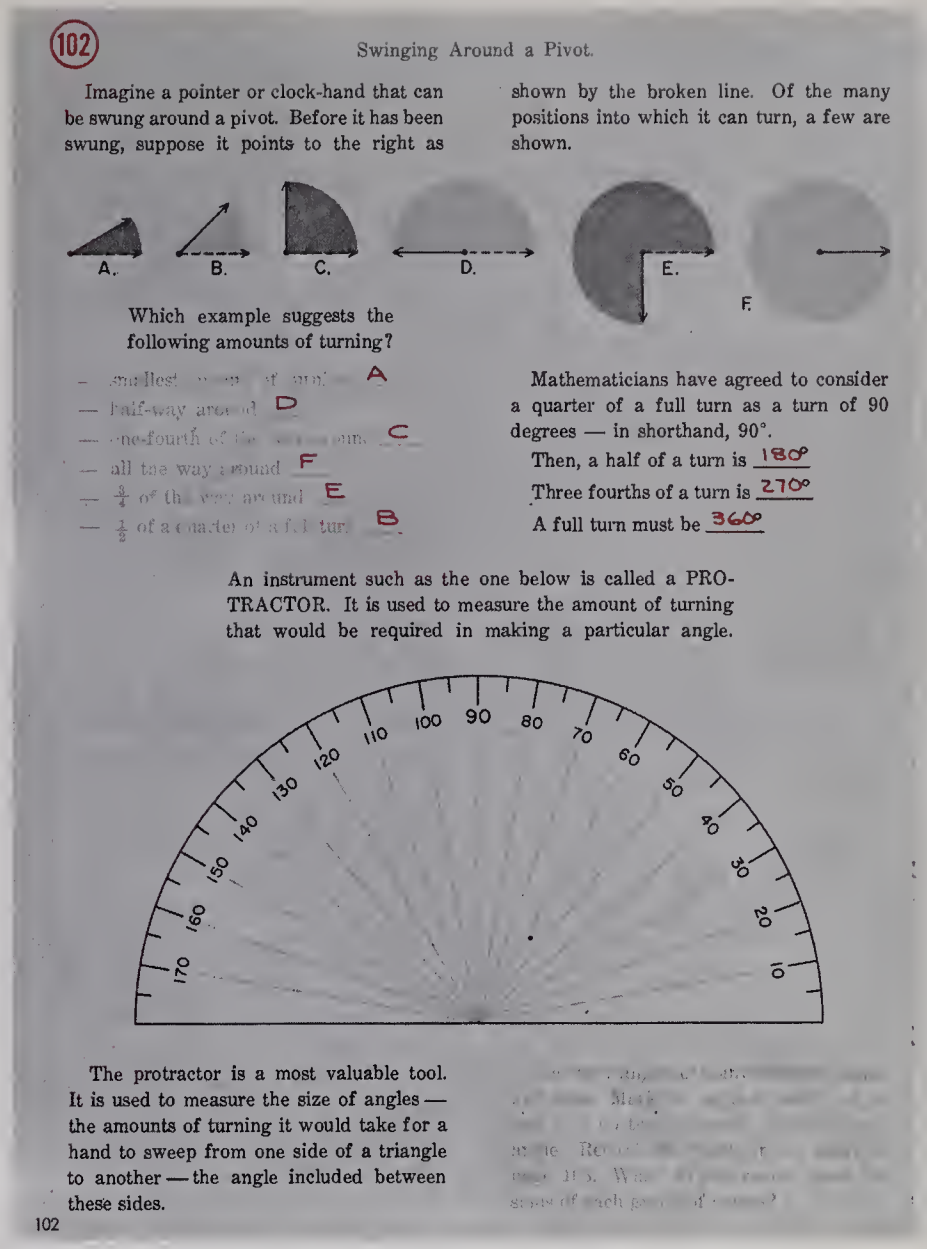
The abbreviation for 90 degrees is 90°, and students should become familiar with this notation from beginning.

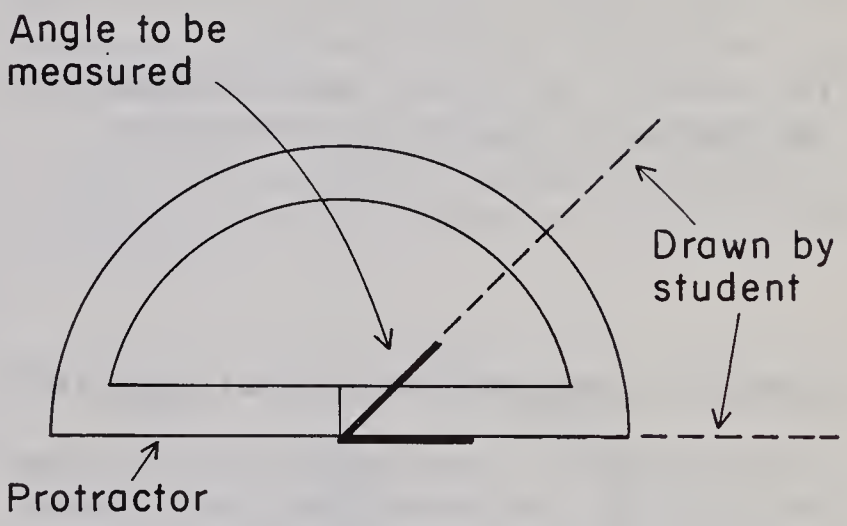
Size of object immaterial

Students need to appreciate that the size of an angle does not depend on the size of an object. The corner of a matchbox, of a piece of paper, of a house, of a football field, of a city block—all represent the same angle: 90°. In each case, the amount of turning is the same as that taken if we begin by facing north and swing around until we are facing east.

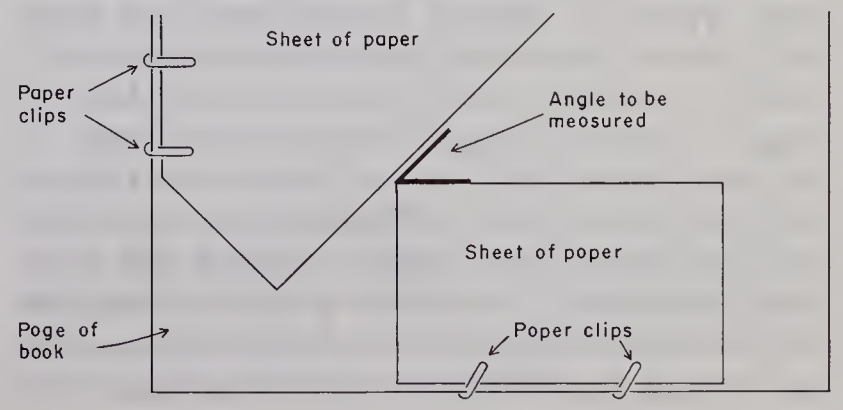


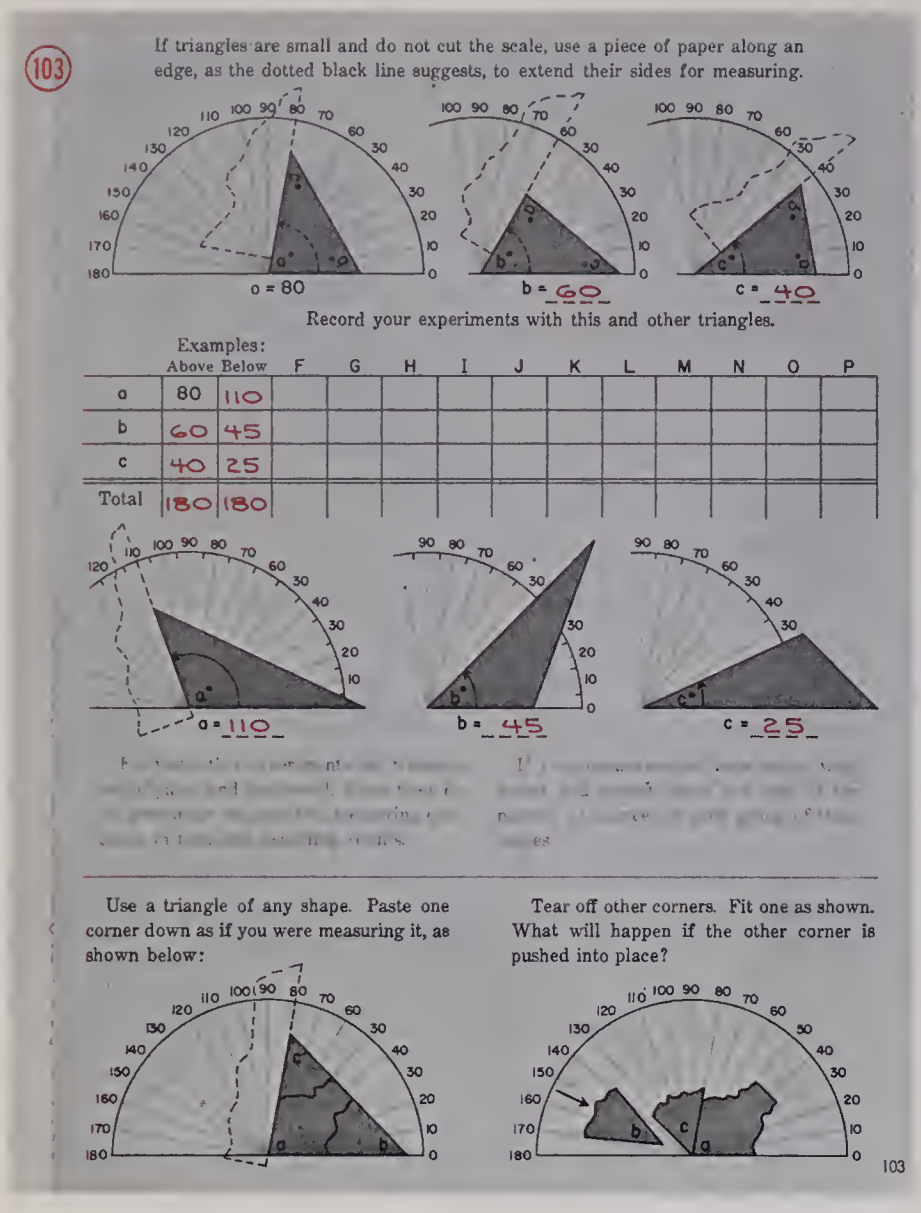
For about a dime, very good protractors can be bought. Some students may own these and wish to use them.


102
Swinging Around a Pivot.
Imagine a pointer or clock-hand that can be swung around a pivot. Before it has been swung, suppose it points to the right as shown by the broken line. Of the many positions into which it can turn, a few are shown.
A.
B.
C.
D.
E.
F.
Which example suggests the following amounts of turning?
— smallest amount of turning A
— half-way around D
— one-fourth of the way around C
— all the way around F
— ¾ of the way around E
— ½ of a quarter of a full turn B
Mathematicians have agreed to consider a quarter of a full turn as a turn of 90 degrees — in shorthand, 90°.
Then, a half of a turn is 180°
Three fourths of a turn is 270°
A full turn must be 360°
An instrument such as the one below is called a PROTRACTOR. It is used to measure the amount of turning that would be required in making a particular angle.
The protractor is a most valuable tool. It is used to measure the size of angles — the amounts of turning it would take for a hand to sweep from one side of a triangle to another — the angle included between these sides.
102



Students may find that an angle they wish to measure is between lines which are too short to reach the rim of the protractor. In such a case, the lines can be extended (as shown by the dotted lines in the diagram). It is also possible to use pieces of paper in order to extend the sides of an angle:




103

If triangles are small and do not cut the scale, use a piece of paper along an edge, as the dotted black line suggests, to extend their sides for measuring.

a = 80 b = 60 c = 40

Record your experiments with this and other triangles.

Examples:

	Above	Below	F	G	H	I	J	K	L	M	N	O	P
a	80	110											
b	60	45											
c	40	25											
Total	180	180											

a = 110 b = 45 c = 25

Use a triangle of any shape. Paste one corner down as if you were measuring it, as shown below:

Tear off other corners. Fit one as shown. What will happen if the other corner is pushed into place?

103

The protractor can then be used to measure the angle between the edges of the pieces of paper.

Notes to accompany *Work Text* page 103

The experiments on this page are intended to show that the angles of a triangle total 180°. Students will make unavoidable errors of measurement, however, and if the totals come between 170° and 190°, this is as much as we can expect.

Although students may be asked to measure the *same* triangle independently, their results will likely differ. Therefore, we can hardly claim that these results prove that the total number of degrees in a triangle is exactly 180°. The teacher will have to assure students that careful measurements made by competent scientists show the angles of a triangle total very close to 180°. In Euclid's geometry, mathematicians study what would follow if this value were *exactly* 180°. The work of Einstein suggests that the actual world differs *very minutely* from the geometry of Euclid. But in surveying, engineering and other practical work, Euclid's geometry is applied. Calculations are made on the assumption that the angles of a triangle total exactly 180°.

Notes to accompany *Work Text* page 104

As on page 103, errors of measurement will occur. The tables can, at best, show that the two angles are *about the same size*. For everyday purposes, it has been found very satisfactory to conclude that these angles are *exactly equal*.

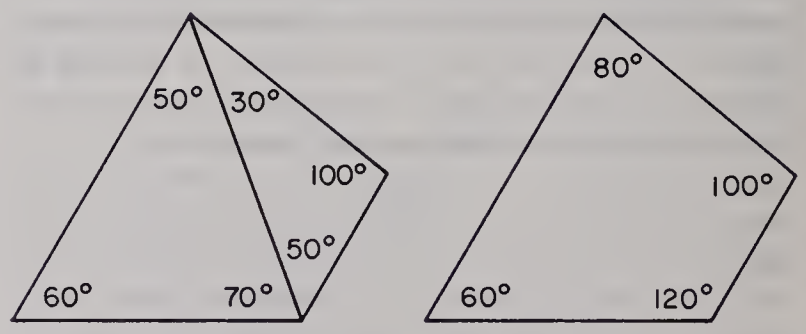
At the bottom of the page, an argument is given to show that if the "Z-result" is true, then the angles of a triangle make exactly 180°.

Notes to accompany *Work Text* page 105

A floor can be covered with tiles in *any* triangular shape. Two such tiles can always be placed in such a way as to form a parallelogram. A series of parallelograms can then be fitted together to cover the floor.

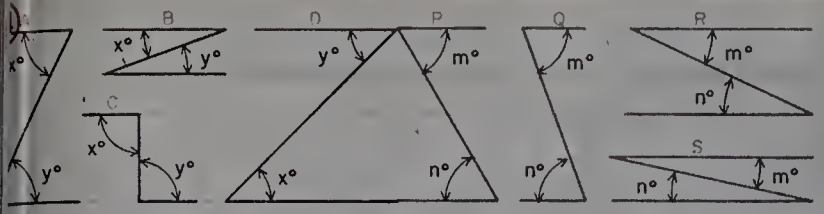
Notes to accompany *Work Text* page 106

The angles of a four-sided figure total 360°.



We can see why this is so if we divide a four-sided figure into two triangles. In the example shown above:

$$180° = 60° + 50° + 70°$$
$$180° = \quad 30° + 50° + 100°$$
$$\text{Add } 360° = 60° + 80° + 120° + 100°$$



In the Z-shapes above, use a protractor to measure angles marked x° and y°. In the backwards Σ-shapes, measure angles marked m° and n°.

On lined paper, draw some Z's of your own — of the front and backward varieties — and record similar measurements of the angles on the chart below.

Number of degrees	A	B	C	D	E	F	G	H	I	J	K
x degrees	63°	20°	90°	45°							
y degrees	63°	20°	90°	45°							

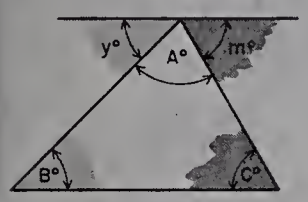
Number of degrees	P	Q	R	S	T	U	V	W	X	Y	Z
m degrees	60°	70°	25°	12°							
n degrees	60°	70°	25°	12°							

In shorthand, we can say very briefly what we notice: x = y and m = n. In the Z and the reverse Σ the angles are the same size.

We call this the "Z-result."

(Turn this page on the side and you might decide to call it the "N-result.")

Below is a copy of the shape or design that appears in the center at the top of the page:



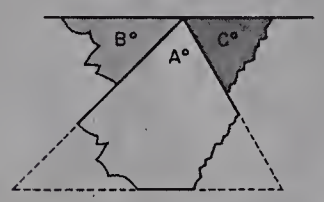
Only the labels have been changed, and a little color has been added.

You can see it as a Z and reverse Σ put together — or as a triangle with a line through one corner.

One Z-result tells us that B = y.

Another Z-result tells us that C = m.

If we cut out the corner marked B° and C° and respectively pasted them on angles marked y° and m°, we would have:

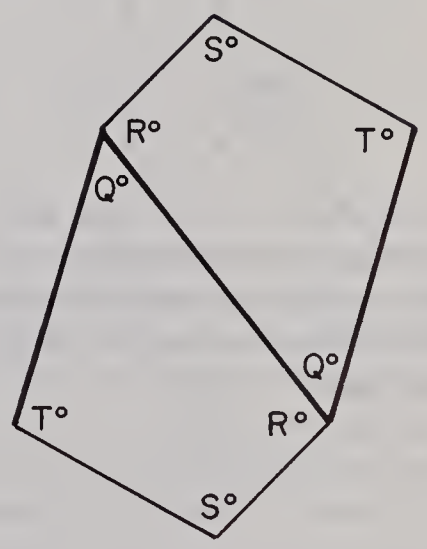


The 50° and the 30° angles joined together form an angle of 80°; the 70° and 50° angles joined together form an angle of 120°. The four angles in the right-hand figure contain as many degrees as there are in the angles of two triangles. One triangle contains a total of 180°, two triangles contain a total of 360°. Therefore, the angles of a four-sided figure also contain a total of 360°. This is the idea used in the second column, which says, "You can reach the same answer by reasoning." In the above illustration, an angle of 80° is broken into 50° and 30°; on *Work Text* page 106, an angle of R° is broken into A° and B°. In the former example, we have presented an argument from arithmetic and used numbers to show that 70° + 60° + 50° = 180°. *Work Text* page 106, however, does not write D° + Q° + A° = 180°. Instead it shows this pictorially: angles of D°, Q° and A° fill one side of a line (and so must make up 180°). In the same way, instead of saying 50° + 100° + 30° = 180°, the text shows angles of C°, S° and B° filling the other side of the line. D° and C° can be joined together to make T° again, and A° and B° can be joined together to make R°.

How to tile a floor

It is possible to make use of the above idea in tiling a floor. At any point where four tiles meet, each of the four angles will be centered. In other words, at any such point there will be one angle of Q°, one of R°, one of S°, and one of T°. It may surprise students that this is possible. Have them experiment in the following way:

Begin with two tiles, cut to the same shape. Arrange the angles of Q° so that they form the inside angles of a Z:



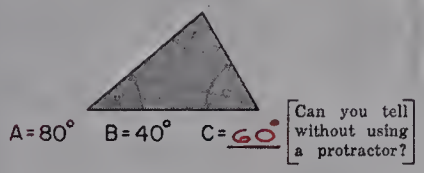
You now have a six-sided figure which has three pairs of parallel sides. Suppose we call this figure a cell, because it is a little like the cell in a honeycomb. Make several of these cells from pairs of tiles. Place them side by side, like this:

105

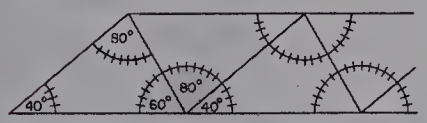
Could you neatly tile a floor using triangular tiles all the same size and shape? We will assume that you can cement pieces down with either side up. Also, you do not need to worry about the corners or the edges of the room.

Draw a triangle on the top sheet of several sheets of paper. Cut as many at a time as possible — so you will have a good supply. (It's more fun if the angles of your pattern triangle are all different sizes.)

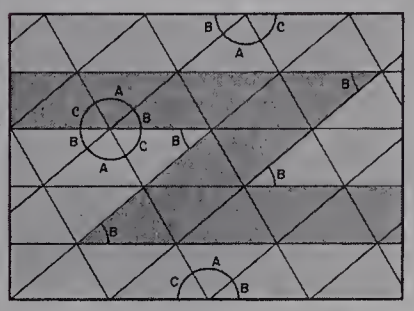
Here is one example:



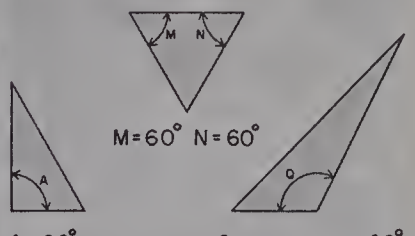
Here is a bridge design:



A pattern for a floor of tile the same shape (but smaller) will look like this, some green and some blue.



Make some floor covering designs of your own. Each time you finish a design, try a different shape. Here are some you might like to choose from:



(Cut out chunks of the patterns you make and paste them in your book on this page.)

You know that a quarter of a circle contains 90 degrees. So, one degree is found by dividing a quarter of a circle into 90 parts — each of which is one degree.

If a quarter of a circle contains 90 degrees, then a half of a circle must contain 90 + 90 or 2 × 90 or 180 degrees.

The experiments in these pages have probably led you to the conclusion that if you tear the three corners off any triangle, and fit them together in a certain way, they will neatly fill half a circle.

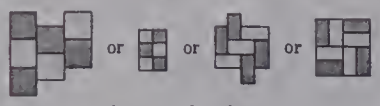
That is a conclusion you can rely upon, and it is a most useful fact to understand.

In shorthand, we write: "If a triangle has angles of sizes A°, B° and C°, then:

$$A° + B° + C° = 180°$$

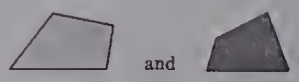
106

Of course you can work out many patterns in tile with a square or with a rectangle — and these are four-sided shapes.



. . . . and many others!

It's harder when all sides are different in length. Try a shape like this:



Start with a large supply of the shape you select. Could you cover a floor with such Tiles?

NOTE: Try your experiment before going on. If you need help later on, you can find hints below.

In experiments, we found that the angles of a triangle add up to 180°. In a four-sided figure, there are four angles. In the example below, what do you think Q° + R° + S° + T° will be?

No. of degrees	Above	Your experiments			
in Q	70				
in R	120				
in S	70				
in T	100				
Total	360				

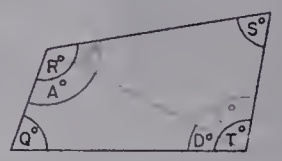
You may want to include a square and a rectangle in your experiments . . . .

Can you finish this shorthand statement: "If a four-sided figure has angles of sizes Q°, R°, S° and T°, then:

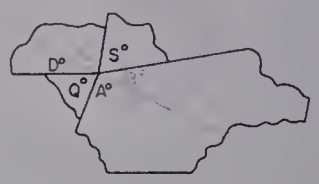
Q + R + S + T = 360°

You can reach the same answer by reasoning — without any measuring.

Draw a line that breaks the figure into two triangles — does this give you any ideas?

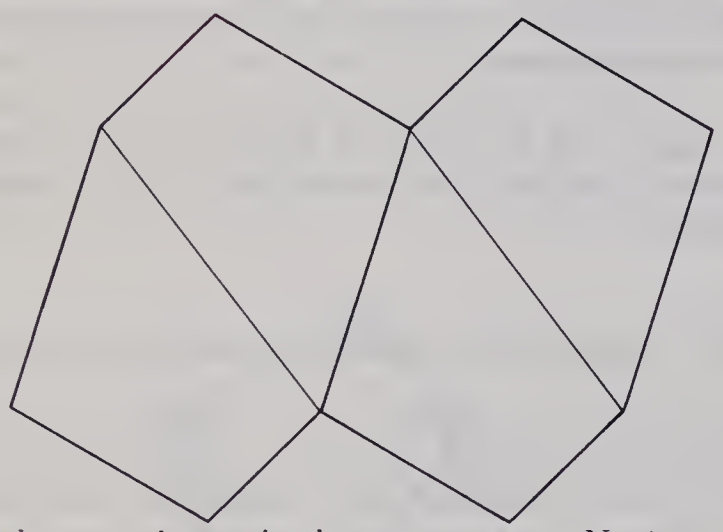


If we tore out three corners and fitted them with the other, we might have:

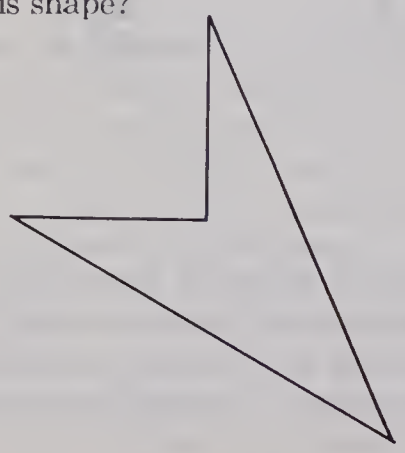


You can tile the floor if you study the diagram below:

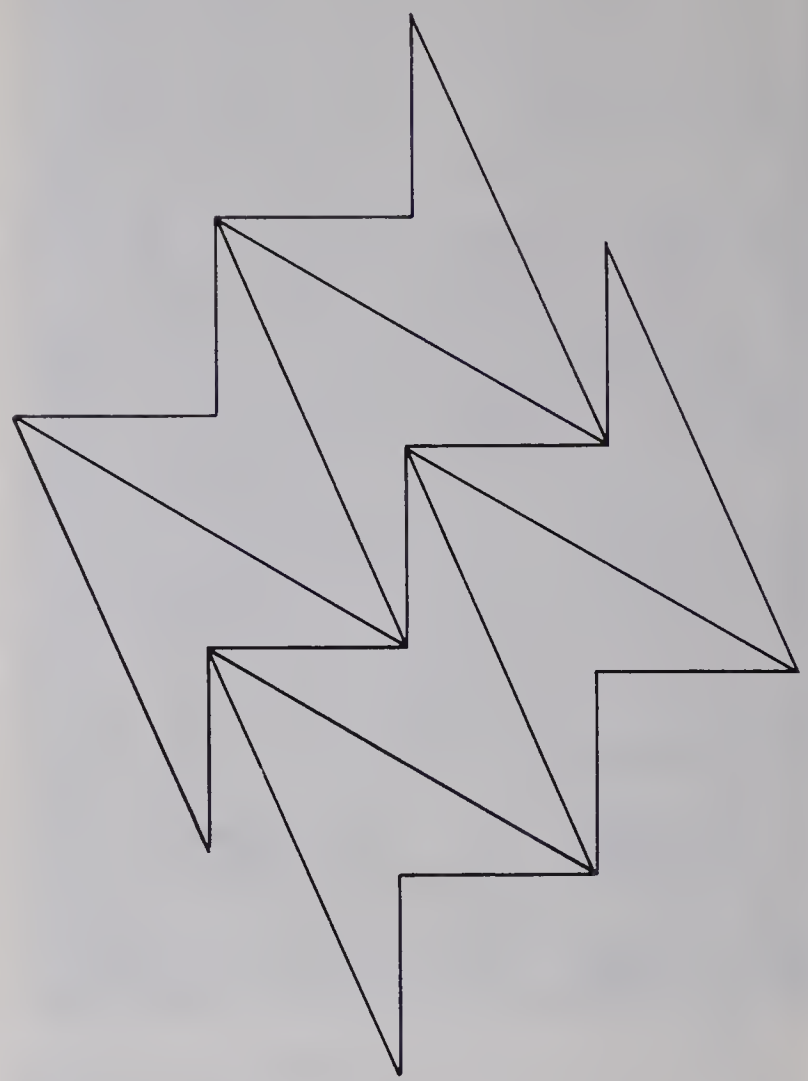
Is this more evidence of your conclusions?



Make several rows in the same manner. Next, push them together so that no spaces are left. Students may ask, "Could you cover a floor if the four-sided tiles were this shape?"



The answer is, "Yes. Exactly the same procedure works." Here is the floor:

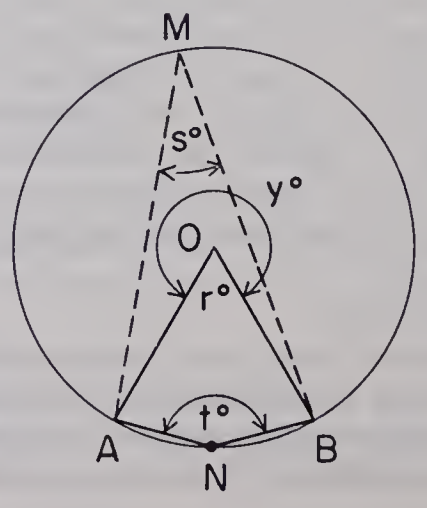


Quite a sophisticated pattern, produced by a mathematical rule.

Notes to accompany *Work Text* page 107

There is relatively little to say about Betty's trick. In Euclid's geometry it is shown that if the angle of r° is twice the angle of s°; and if it is also twice the angle of t°, then s and t must be the same number.

Note that the points M and N must be placed between A and B, as in this figure:



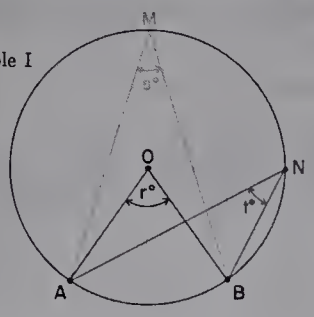
Betty ... does it work" • Why? • Does it always work?

(Study this trick ... your own or with another student. Find out how much you can learn through your own experiments.)

"I learned a nice trick from my brother," Betty announced. "He is studying geometry. The trick goes like this:

"Draw a circle any size and number the center with the letter 'O.' I want Harry to choose two points on the circle and label them A and B. Connect A to O and B to O.

Example I



"Mark the angle formed as r°.

"Now I want Ruth to find another point on the circle and label it M.

"I want Bob to find another point on the circle and label it N.

"Draw lines from A and B to M. Label the angle formed as s°. Draw lines from A and B to N. Label the angle formed as t°.

"Now," Betty reminded them, "I had Harry and Ruth and Bob do the drawing.

"I want Harry to use a protractor and measure the angle marked r°." Harry did. "It's about 70°," he said.

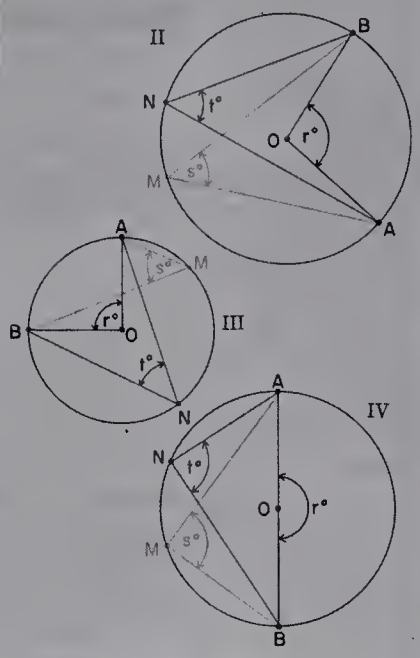
"Now Ruth can measure the angle marked s° and Bob can measure the angle marked t°. Unless Ruth or Harry or Bob make a mistake, Ruth will find that s = 35 and Bob will find that t = 35."

Bob and Ruth took turns with the protractor.

"Betty's right," Bob exclaimed.

"She certainly is," agreed Ruth.

"Let's try it again," Harry urged. They did. Here are some of their results. Record what you find below.



Number of Degrees

	r	s	t
I	70	35	35
II	100	50	50
III	90	45	45
IV	180	90	90

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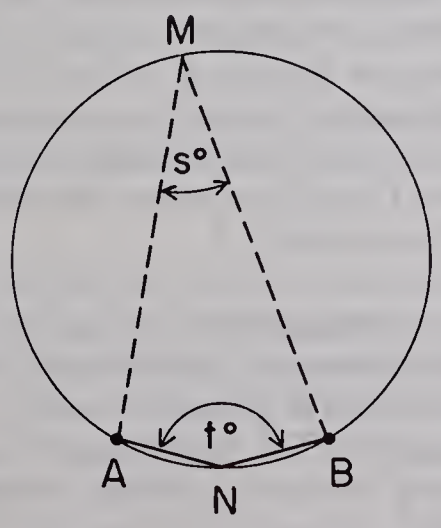
y° would be twice t°, and the size of t would *not* be the same as the size of s. For example, suppose r° were 60° Then s° would be 30°, since 30 $= \frac{1}{2} \times 60$. But y° would be 300°, since a complete revolution contains 360°, and y° and r° together make up a complete revolution. If y° is 300°, t° must be half of that, so t° would be 150°. 150 for t is not the same as 30 for s. However, you may notice that t + s = 150 + 30 = 180. This always happens whenever M and N are taken, so to speak, on opposite sides of AB:



In this figure, s° + t° = 180°. You may, however, not wish to go into this with your students.

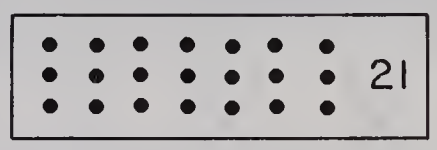
General notes to accompany *Work Text* pages dealing with algebra

In Grade 4 (*Level D*), a section at the end of the text was prepared as an introduction to algebra for students ready for it. Therefore, some students in Grade 5 will need only to be reminded of what they did last year.

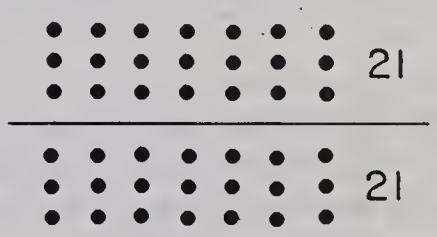
For others, however, algebra may be something entirely new. It may be useful to glance at the Grade 4 *Work Text* and *Teacher's Guide* in order to see how the ideas of algebra were introduced there.

The transition from arithmetic to algebra should be imperceptible. The student should not be able to say, "At this page I left arithmetic and started algebra," because these subjects are essentially one.

Consider an example. Suppose a student has forgotten what six sevens are. The teacher might say, "Do you know three sevens?" "Yes, 21." The teacher might then draw three rows of seven dots each on the board. Beside it she would write 21. The teacher might then ask, "What would you do to change this into a picture of *six* sevens?"



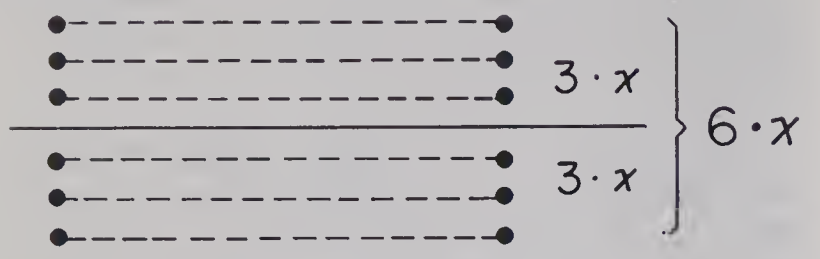
The student would be expected to draw three more rows of seven dots each and (at the teacher's suggestion) write 21 beside the drawing. The chalkboard would now appear like this:



If the teacher were then to ask, "Now can you tell me what six sevens are?" the student could reason, "I see that six sevens are 21 + 21, which makes 42," or perhaps, "Six sevens are twice 21, so they must make 42." All this is within the realm of arithmetic. But it might occur to a student that it would be possible to use this idea to find *any* result in the six times table. When a student says, "If you take any number, multiply it by 3 and double the result, the answer will be 6 times the number," he is making a common-sense statement about arithmetic. The statement also pertains to algebra, because it refers to *an unknown number*—any number. In algebra this statement may be expressed in a special shorthand, but the shorthand is only a device. The important aspect of algebra is its applica-

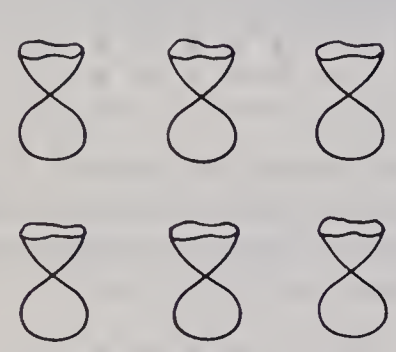
tion to general statements that are true. With regard to the statement, "If you double 3 times any number, you get 6 times that number," it is a very small step to introduce x as an abbreviation for "any number" and · as an abbreviation for "times". The statement is then expressed in algebra shorthand, "If you double 3 · x, you get 6x."

Using the picture of six sevens we can turn it into a picture of six times *any number*:



The same type of illustration will be found on *Work Text* page 120.

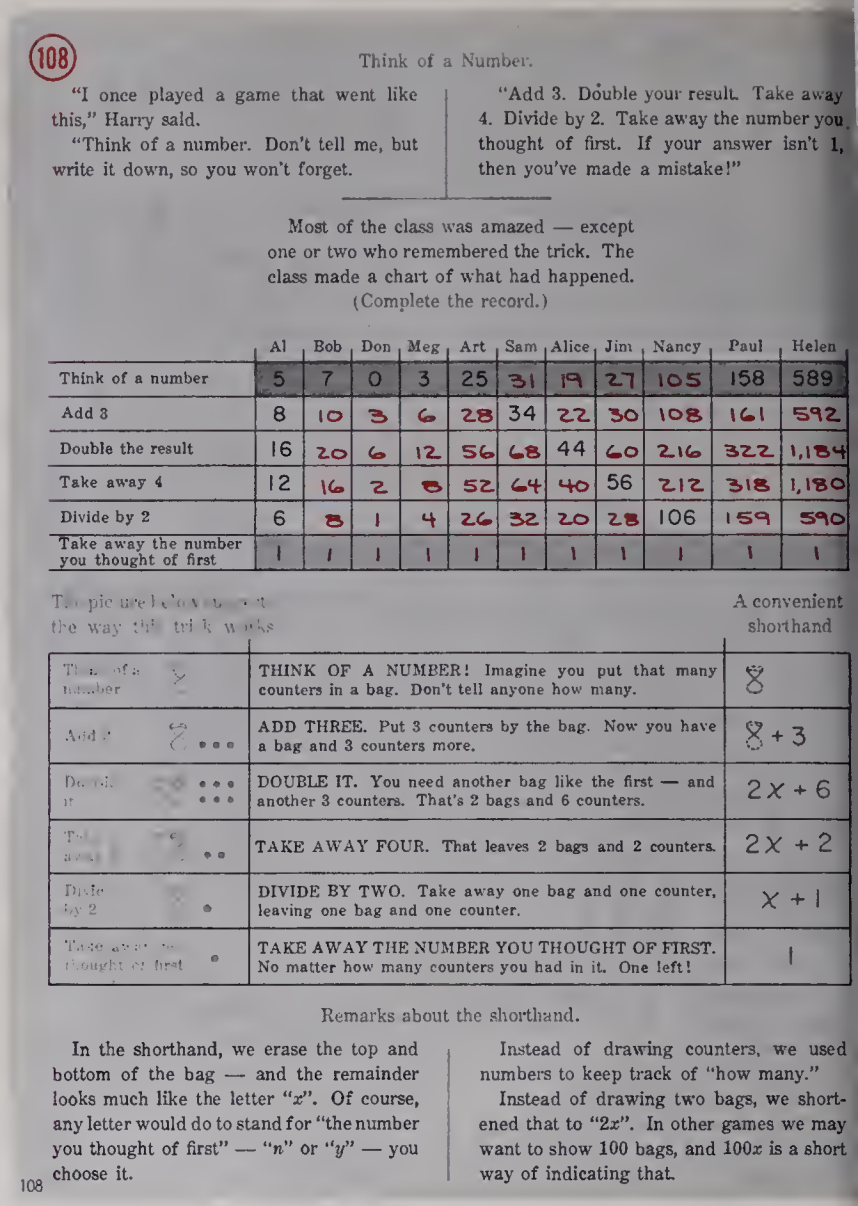
It is also possible to illustrate the above statement in the following way: we draw 3 bags. We say each bag has the *same* unknown number of counters in it. (The number can be *any number*). We then draw 3 more bags. These, too, each contain the same number of counters.



We can say we have "three bags twice" or "six bags." The picture demonstrates that "three times a number, doubled" and "six times a number" are two ways of saying the same thing. The "bag" picture is used on *Work Text* page 108.

Notes to accompany *Work Text* page 108

In the columns headed *Bob*, *Don*, *Meg*, *Art*, *Paul* and *Alice*, students are given the numbers thought of and are asked to carry out the instructions of the trick. Students will observe that the answer is 1 in each case. But in the columns headed *Nancy*, *Helen*, *Jim* and *Sam*, students are only given numbers that occur during the steps involved in the trick. They are not given the numbers thought of. *It is perfectly satisfactory for students to complete the trick by any method they choose.* Most students will probably work backwards: "After adding 3 to her number, Nancy got 34; she must have started with 31." "Helen had 44 after doubling; so, she must have had 22 before doubling; before adding 3 she must have had 19." In the same way students can work out Jim's and Sam's columns.



108

Think of a Number.

"I once played a game that went like this," Harry said.

"Think of a number. Don't tell me, but write it down, so you won't forget.

"Add 3. Double your result. Take away 4. Divide by 2. Take away the number you thought of first. If your answer isn't 1, then you've made a mistake!"

Most of the class was amazed — except one or two who remembered the trick. The class made a chart of what had happened. (Complete the record.)

	Al	Bob	Don	Meg	Art	Sam	Alice	Jim	Nancy	Paul	Helen
Think of a number	5	7	0	3	25	31	19	27	105	158	589
Add 3	8	10	3	6	28	34	22	30	108	161	592
Double the result	16	20	6	12	56	68	44	60	216	322	1,184
Take away 4	12	16	2	8	52	64	40	56	212	318	1,180
Divide by 2	6	8	1	4	26	32	20	28	106	159	590
Take away the number you thought of first	1	1	1	1	1	1	1	1	1	1	1

The picture below shows the way the trick works		A convenient shorthand
Think of a number	THINK OF A NUMBER! Imagine you put that many counters in a bag. Don't tell anyone how many.	$\square$
Add 3	ADD THREE. Put 3 counters by the bag. Now you have a bag and 3 counters more.	$\square + 3$
Double it	DOUBLE IT. You need another bag like the first — and another 3 counters. That's 2 bags and 6 counters.	$2x + 6$
Take away 4	TAKE AWAY FOUR. That leaves 2 bags and 2 counters.	$2x + 2$
Divide by 2	DIVIDE BY TWO. Take away one bag and one counter, leaving one bag and one counter.	$x + 1$
Take away the number you thought of first	TAKE AWAY THE NUMBER YOU THOUGHT OF FIRST. No matter how many counters you had in it. One left!	1

Remarks about the shorthand.

In the shorthand, we erase the top and bottom of the bag — and the remainder looks much like the letter "x". Of course, any letter would do to stand for "the number you thought of first" — "n" or "y" — you choose it.

Instead of drawing counters, we used numbers to keep track of "how many."

Instead of drawing two bags, we shortened that to "$2x$". In other games we may want to show 100 bags, and $100x$ is a short way of indicating that.

Notes to accompany *Work Text* page 109

Students may give the answer 1 + x for "one more than the number thought of," where we have given x + 1. Both 1 + x and x + 1 are correct. If some students have x + 1 and others have 1 + x, it is good to let the students discuss the question before giving the verdict. It may be a good idea to test particular examples. "Suppose someone thought of the number 8. Then to him 1 + x would suggest 1 + 8 and x + 1 would suggest 8 + 1. Since 1 + 8 and 8 + 1 are both ways of writing 9, it would not matter which expression was used." Some students may see at once that it doesn't matter whether you add 1 to the number or the number to 1.

The above remarks apply to the fact that 3 + x and x + 3 have the same meaning; and that 2 + x is as good as x + 2. On the other hand, 2 − x is *not* the same as x − 2. The first means "subtract your number from 2" while x − 2 means "subtract 2 from your number."

Omission of multiplication sign

It usually takes time for students to get used to the

fact that in algebra the multiplication sign is omitted. There are several ways of dealing with this. It is possible to point out that it is natural for us to say "three bags" rather than "three times a bag." Even in arithmetic, it is not unusual to say "three sevens" rather than "three times seven."

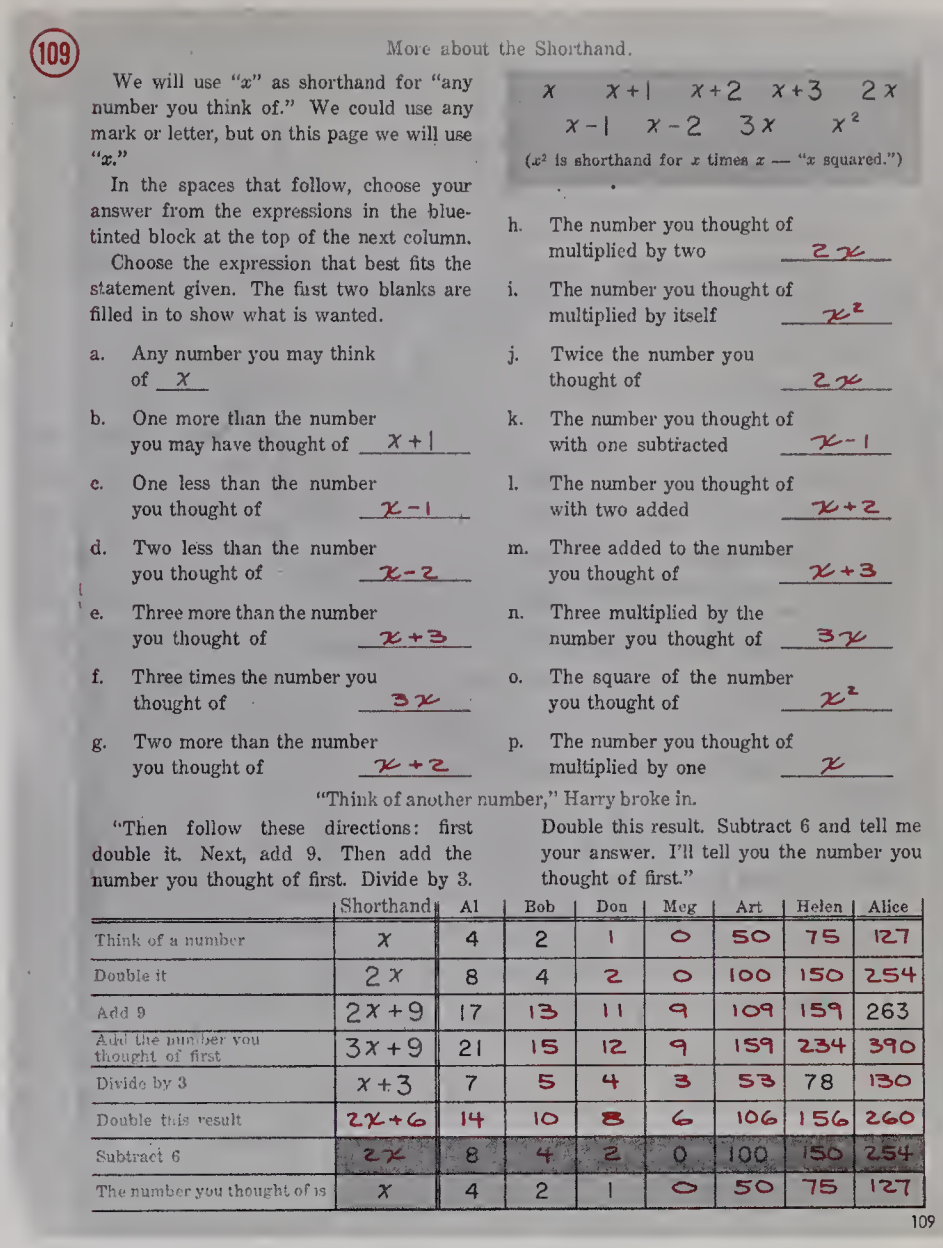
It is also possible to introduce some form of transition. We can start by writing a small multiplication sign: 3✖ x, and then add a dot in the middle so that sign looks like a Sputnik: 3✖ x. Then, we can agree to keep the dot only: 3 · x, thereby employing a form used in serious mathematical work. Finally, we can omit the dot and reach the more usual form: 3x.

At this stage, we accept x·3 or x3 as perfectly good equivalents of 3·x or 3x. It is *usual* in algebra to write 3x rather than x3, but there is no error of thought in writing x·3 for "your number times 3." It is more natural to say "3 times your number," but this is merely a matter of convention. Our aim is to teach children to think logically, so we accept any answer that is logical. Experience will teach students that it is customary to write 3x rather than x3, and they will probably slip into this habit without even being aware of it. No principle is involved.

In question *p*, we give the answer simply as x. However, x·1 and 1·x and 1x and x1 must also be accepted as perfectly correct. If we think of x as "bag" then 1x can be read as "one bag" and this is a correct answer to the question. Again, it is the *custom* to write x rather than 1x or 1·x, but we do not want to emphasize customs. We want to emphasize *logical thinking* and all the answers given above are logically correct.

Students usually need to be reminded many times *not* to read and *not* to think of 3x as "3 and x." Pictures may help. How do we picture x? As a bag. How do we picture 3x? As three bags. Do these bags represent three *added to* a bag? No. If there were 8 counters in each bag, three bags would represent 24 counters, or 3 *times* 8. Such drilling will have to be repeated many, many times before every student absorbs the idea.

In subtraction, the order of writing can be troublesome; students are usually bothered by the fact that $5 - 2$ means "take 2 from 5". The order of writing is easier to grasp if students will apply such phrases as, "5 less 2" or "5, take away 2." If students are perfectly clear about $5 - 2$ and $8 - 2$ and $13 - 2$, they will understand $x - 2$, "the number you thought of, take away 2" (or "less 2" or "reduced by 2" or any other words students find helpful).



109

More about the Shorthand.

We will use "x" as shorthand for "any number you think of." We could use any mark or letter, but on this page we will use "x."

In the spaces that follow, choose your answer from the expressions in the blue-tinted block at the top of the next column.

Choose the expression that best fits the statement given. The first two blanks are filled in to show what is wanted.

x $x+1$ $x+2$ $x+3$ $2x$
$x-1$ $x-2$ $3x$ x^2

(x^2 is shorthand for x times x — "x squared.")

a. Any number you may think of x

b. One more than the number you may have thought of $x+1$

c. One less than the number you thought of $x-1$

d. Two less than the number you thought of $x-2$

e. Three more than the number you thought of $x+3$

f. Three times the number you thought of $3x$

g. Two more than the number you thought of $x+2$

h. The number you thought of multiplied by two $2x$

i. The number you thought of multiplied by itself x^2

j. Twice the number you thought of $2x$

k. The number you thought of with one subtracted $x-1$

l. The number you thought of with two added $x+2$

m. Three added to the number you thought of $x+3$

n. Three multiplied by the number you thought of $3x$

o. The square of the number you thought of x^2

p. The number you thought of multiplied by one x

"Think of another number," Harry broke in. "Then follow these directions: first double it. Next, add 9. Then add the number you thought of first. Divide by 3. Double this result. Subtract 6 and tell me your answer. I'll tell you the number you thought of first."

	Shorthand	Al	Bob	Don	Meg	Art	Helen	Alice
Think of a number	x	4	2	1	0	50	75	127
Double it	$2x$	8	4	2	0	100	150	254
Add 9	$2x+9$	17	13	11	9	109	159	263
Add the number you thought of first	$3x+9$	21	15	12	9	159	234	390
Divide by 3	$x+3$	7	5	4	3	53	78	130
Double this result	$2x+6$	14	10	8	6	106	156	260
Subtract 6	$2x$	8	4	2	0	100	150	254
The number you thought of is	x	4	2	1	0	50	75	127

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Exponents

Two questions involve x^2, "the square of the number you thought of." Students frequently want to read 10^2 as "twice ten." This difficulty belongs to *arithmetic, not to algebra.* One should not allow students to embark on the algebra of x^2 and x^3 and x^5 until they are perfectly clear that 2^5 means $2x2x2x2x2 = 32$, not $2x5 = 10$; and that 2^3 is 8, not 6.

If students cannot answer immediately such questions as, "What does 10^3 mean?" and "What is 3^2?" it would be wise to review the idea before dealing with questions *i* and *o* on this page.

However, even if students do not understand exponents, it is perfectly satisfactory to proceed with the other questions. The central idea of algebra is that you can use "a bag" or the letter x to represent "any number someone has thought of." This is an interesting new idea and students should be allowed to explore it. Exponents are not the central idea of algebra. They can be brought in later. Once a student realizes that 10^3 means 10 times 10 times 10, and 7^3 means 7 times 7 times 7, he will not find it hard to realize that x^3 means the number thought of times itself three times.

It is not until *Work Text* page 119 that students actually begin to use x^2 in their work. Two or three

minutes a day might be devoted to questions such as: "What is a short way of writing 7x7x7x7x7?" (Answer: 7^5) "What is the meaning of 10^3?" (Answer: 10x10x10, or "a thousand") "What is 5^2?" (Answer: 5x5, or 25) This should be sufficient to remind students of the meaning of exponents.

Approach to examples at bottom of page

The work at the bottom of the page can be helped by pictures of bags. For example, in the fourth row we have 3x + 9, which we may picture as 3 bags and 9 counters. The next instruction is, "Divide by 3." If 3 bags and 9 counters are to be divided equally among 3 children, how much would each child get? Answer: a bag and 3 counters; shorthand: x + 3. In the same way, the other steps can be worked out with the help of pictures.

In the row before the last, we see 2x under the *shorthand* column. This means that the trick at this stage always gives *twice* the number first thought of. *So, to tell the original number, we divide by 2.* Notice that the last row gives x, the number thought of. Don has 1 in the last row—so he began with 1, and we can work out his column from this. Meg has 0 in the row labeled 2x. If twice her number is zero, her number must be zero. Art has 100 in the row for 2x. So, his number must have been 50.

It is possible to fill Helen's column in two ways. If 78 corresponds to "a bag and 3 counters," the bag must hold 75 counters. This, then, is Helen's number and the rest of the column can be worked out on the basis of this. Or it is possible to work down the column, as the trick directs. Double 78; this gives 156. Subtract 6; this gives 150 for the row, 2x. So "2 bags" hold 150 counters. Then 1 bag must hold 75.

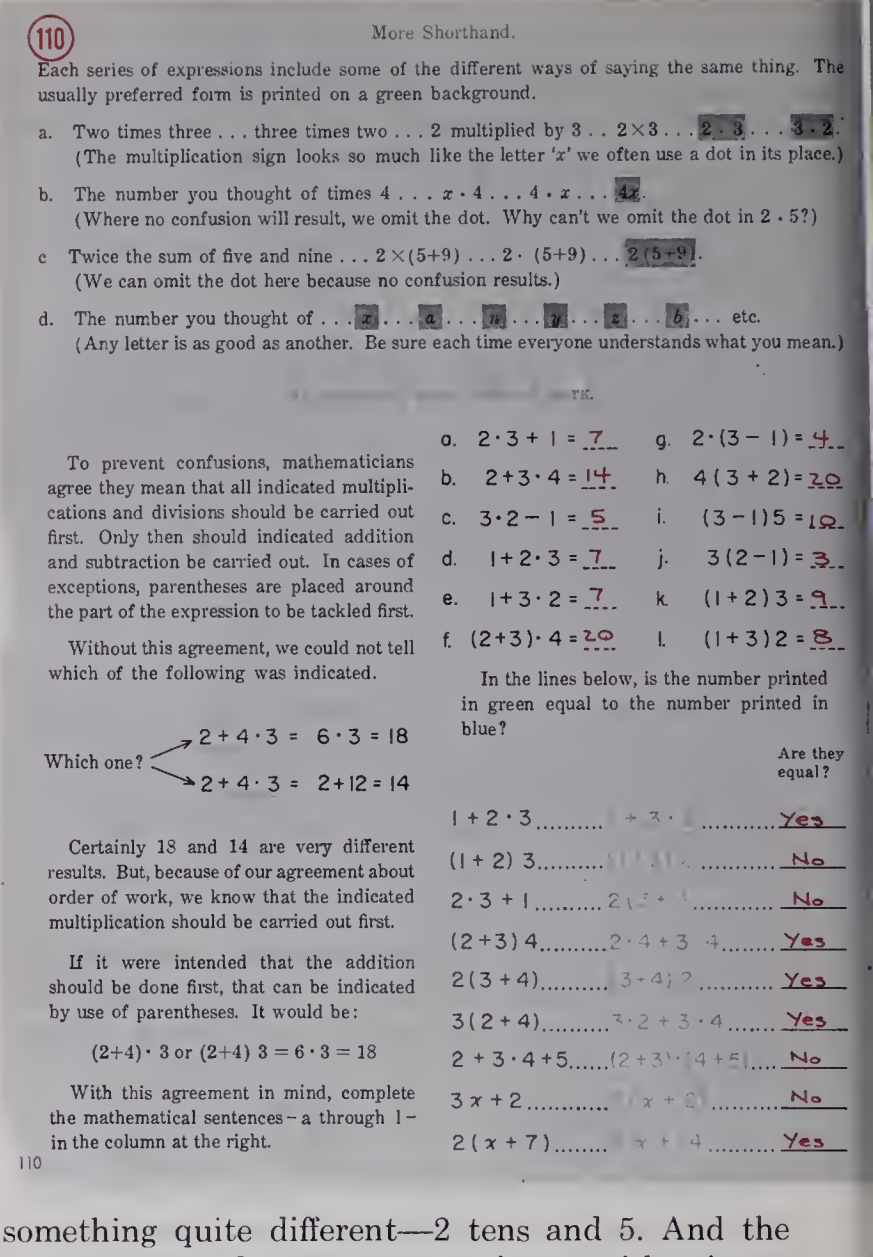
We cannot work down the column for Alice, because the instructions say, "add the number first thought of" and we do not yet know what she first thought of. But we can work up the column. After adding 9, she had 263. Before adding 9, she must have had 254. Before doubling, she must have had 127. So, Alice began with 127.

Notes to accompany *Work Text* page 110

In *b*, we cannot omit the dot (for *times*) that appears between the 2 and 5 (2·5) because 25 would be read as "twenty-five." Children often point out that this is illogical. And in a sense they are right. In algebra, when we put two symbols such as 2 and x together to make 2x, we want the notation to mean multiplication. In arithmetic, when we put 2 and 5 together as in 25, we want the notation to mean something quite different—2 tens and 5. And the trouble is, we do not want to give up either interpretation! We still want 25 for writing "twenty-five," because it would be very hard to find a new way of writing "twenty-five." And we still want 2x to stand for "two times the number thought of." The students may want to try to invent a more logical system. Until they, or someone else does, we shall have to be illogical and write 25x for "25 bags" or "25 times the number thought of."

Using parentheses

The rest of this page deals with parenthesis. What does a waiter mean when he orders two eggs and bacon from the kitchen? Does he mean that one customer wants bacon with two eggs, or that two customers each want bacon and one egg? We have the same difficulty in arithmetic with "twice three and four." Does this mean "twice seven" or "six and four." We use parentheses ( ) to make the distinction. Thus, 2· (3 + 4) means "twice three and four," while 2 · 3 + 4 means two threes and and a single four. Pictures of the following type may be helpful:



110 More Shorthand.

Each series of expressions include some of the different ways of saying the same thing. The usually preferred form is printed on a green background.

a. Two times three . . . three times two . . . 2 multiplied by 3 . . 2×3 . . . 2 · 3 . . . 3 · 2.
(The multiplication sign looks so much like the letter 'x' we often use a dot in its place.)

b. The number you thought of times 4 . . . $x \cdot 4$ . . . $4 \cdot x$ . . . $4x$.
(Where no confusion will result, we omit the dot. Why can't we omit the dot in 2 · 5?)

c Twice the sum of five and nine . . . 2×(5+9) . . . 2· (5+9) . . . 2(5+9).
(We can omit the dot here because no confusion results.)

d. The number you thought of . . . x . . . a . . . n . . . y . . . z . . . b . . . etc.
(Any letter is as good as another. Be sure each time everyone understands what you mean.)

To prevent confusions, mathematicians agree they mean that all indicated multiplications and divisions should be carried out first. Only then should indicated addition and subtraction be carried out. In cases of exceptions, parentheses are placed around the part of the expression to be tackled first.

Without this agreement, we could not tell which of the following was indicated.

Which one? → 2 + 4 · 3 = 6 · 3 = 18
Which one? → 2 + 4 · 3 = 2 + 12 = 14

Certainly 18 and 14 are very different results. But, because of our agreement about order of work, we know that the indicated multiplication should be carried out first.

If it were intended that the addition should be done first, that can be indicated by use of parentheses. It would be:

(2+4)· 3 or (2+4) 3 = 6 · 3 = 18

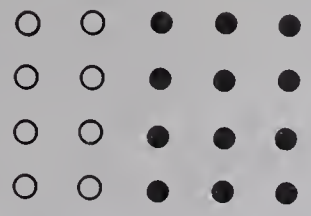
With this agreement in mind, complete the mathematical sentences – a through l – in the column at the right.

a. 2·3 + 1 = 7
b. 2+3·4 = 14
c. 3·2 − 1 = 5
d. 1+2·3 = 7
e. 1+3·2 = 7
f. (2+3)·4 = 20
g. 2·(3 − 1) = 4
h. 4(3 + 2) = 20
i. (3 − 1)5 = 10
j. 3(2 − 1) = 3
k. (1 + 2)3 = 9
l. (1 + 3)2 = 8

In the lines below, is the number printed in green equal to the number printed in blue?

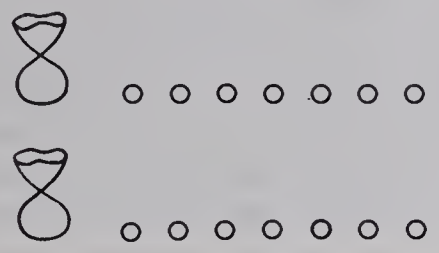
	Are they equal?
1 + 2 · 3 ..... + 3 ·	Yes
(1 + 2) 3	No
2 · 3 + 1 ..... 2 (+	No
(2 + 3) 4 ..... 2 · 4 + 3 4	Yes
2(3 + 4) ..... 3 + 4) ?	Yes
3(2 + 4) ..... 3 · 2 + 3 · 4	Yes
2 + 3 · 4 + 5 ..... (2 + 3 · (4 + 5)	No
3x + 2 ..... (x + 2)	No
2(x + 7) ..... x + 4	Yes

Ask students to draw pictures to illustrate $3 \cdot 5 + 4$ and $3\ (5 + 4)$. You might also draw pictures such as:



and ask students to supply the label. Here, $4 \cdot (2 + 3)$ is correct and $4 \cdot 2 + 3$ is wrong. (There are other correct answers, such as $4 \cdot 5$ or $4 \cdot 2 + 4 \cdot 3$.)

In the last two questions on this page, algebra is involved. Is $3x + 2$ the same as $3\ (x + 2)$? Draw pictures. $3x + 2$ corresponds to 3 bags and 2 counters. $3\ (x + 2)$ means that a bag and 2 counters are *each* multiplied by 3. In the last question, $2\ (x + 7)$ means that a bag and 7 counters are each multiplied by 2.



This is indeed 2 bags and 14 counters, or $2x + 14$. So $2\ (x + 7)$ and $2x + 14$ are different ways of describing the same thing. Have students test this statement by thinking of a number—$2\ (x + 7)$ corresponds to "think of a number; add 7; double." $2x + 14$ corresponds to "think of a number; double it; add 14."

Notes to accompany *Work Text* page 111

First trick: The shorthand can be thought out by drawing pictures of bags. When students have followed the instruction, "divide by 4—and tell your result," the shorthand shows $x + 4$. That is, the number announced is 4 more than the number first thought of. So the person doing the trick simply subtracts 4 from the number announced.

All we know about Bill is that he announced 4. So he must have started with zero. Nan announced 14. She must have started with 10.

We cannot work down Doris' column, for we do not know how to deal with the instruction, "add the number first thought of." But we can easily work backwards. After tripling, she had 93. Therefore, before tripling she must have had 31. Before adding 10, she must have had 21. Or we could argue: 3 bags and 30 counters make 93 counters. So 3 bags must make 63. One bag must contain 21 counters. So she thought of 21.

With Jack we can only work backwards, up the column, or use the argument: If "3 bags and 16 make 190," then 3 bags make 174; and 1 bag makes 58.

With Fay we have to work down: 18 "divided by 4" gives $4\frac{1}{2}$, which Fay announces. So Fay must have begun with $\frac{1}{2}$. (See first paragraph above.)

With Ken, work backwards, or use the same method used for Doris.

Second trick: Here *a* stands for the number thought of, or the number of counters in a bag. The most primitive method is simply to work backwards through all the steps (for no step mentions the unknown number first thought of). A quicker method is to use the shorthand $2a - 1$. The number announced is 1 less than twice the number first thought of.

(111)

More Examples of Harry's "Think of a Number Trick:"

	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
	x	2	0	10	21	58	$\frac{1}{2}$	1.8
	$x + 10$	12	10	20	31	68	$10\frac{1}{2}$	11.8
	$3x + 30$	36	30	60	93	204	$31\frac{1}{2}$	35.4
	$3x + 16$	22	16	46	79	190	$17\frac{1}{2}$	21.4
	$4x + 16$	24	16	56	100	248	18	23.2
	$x + 4$	6	4	14	25	62	$4\frac{1}{2}$	5.8
	x	2	0	10	21	58	$\frac{1}{2}$	1.8

	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
	a	3	5	1	10	.5	$1\frac{1}{4}$	$\frac{5}{8}$
	$a + 2$	5	7	3	12	2.5	$3\frac{1}{4}$	$2\frac{5}{8}$
	$6a + 12$	30	42	18	72	15	$19\frac{1}{2}$	$15\frac{3}{4}$
	$6a$	18	30	6	60	3	$7\frac{1}{2}$	$3\frac{3}{4}$
	$2a$	6	10	2	20	1	$2\frac{1}{2}$	$1\frac{1}{4}$
	$2a - 1$	5	9	1	19	0	$1\frac{1}{2}$	$\frac{1}{4}$
	a	3	5	1	10	.5	$1\frac{1}{4}$	$\frac{5}{8}$

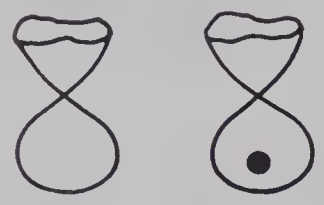
	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
	n	10	0	7	15	25	.1	1.07
	$4n$	40	0	28	60	100	.4	4.28
Add 6	$4n + 6$	46	6	34	66	106	6.4	10.28
	$2n + 3$	23	3	17	33	53	3.2	5.14
	n	10	0	7	15	25	.1	1.07

*Be careful! This one is different! Some of the directions are missing.

	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
	y	10	0	1	.3	7	$\frac{1}{2}$	5
Add 3	$y + 3$	13	3	4	3.3	10	$3\frac{1}{2}$	8
	$100y + 300$	1,300	300	400	330	1,000	350	800
Subtract 50	$100y + 250$	1,250	250	350	280	950	300	750
	$4y + 10$	50	10	14	11.2	38	12	30
	$4y + 4$	44	4	8	5.2	32	6	24
	y	10	0	1	.3	7	$\frac{1}{2}$	5

111

You may want to draw 2a − 1. It is usually good to accompany work in algebra with the pictures of bags. How do we show two bags *less* one counter? Here is a possible method:



A hole has been cut in a bag, and 1 counter removed. (If 3 counters had been removed, we would draw 3 holes.)

For Pete's column, it is possible to argue: two bags with one counter removed contain five counters. Let us restore the missing counter. Two bags then contain 6 counters. So each bag contains 3 counters. Pete thought of 3.

The same method can be applied to the next four columns.

For Fay, students only have to work back one step to find the number she thought of. Then it is possible for them to work down her column.

For Ken, students are told what he started with. There is no problem here, but a drill on fractions has been provided for.

Third trick: For Pete, the easiest way is to argue: "Two bags and 3 counters make 23 counters: each bag must hold 10 counters." Same for Bill, Nan, Doris. For Jack, "4 bags and 6 make 106. 4 bags must make 100," etc. For Fay, "4 bags contain .4 counters. Each bag must contain .1." Ken, we are told, thought of 1.07. No problem; just a drill on decimals.

Fourth trick: We notice:

	Doris	*Ken*
Think of a number	.3	5
	3.3	8

Not much evidence here! But "add 3" is suggested for the missing instruction. We can now fill in the following on our table:

	Shorthand	Pete	Bill	Nan	Doris	Jack	Fay	Ken
Think of a number	y	10	0	-----	.3	7	$\frac{1}{2}$	5
		↓	↑			↑	↑	
Add 3	$y+3$	13	3	-----	3.3	10	$3\frac{1}{2}$	8
		↓	↑		↓	↑	↓	↓
Multiply by 100	$100y + 300$	1300	300	-----	330	1000	350	800
----------			250					750
			↑					
Divide by 25			10					

The arrows show how we work from one number to another. For example, Bill got 10 after dividing by 25. So the number above 10 must be 250.

Looking at Bill's and Ken's columns suggests that the missing instruction is "subtract 50."

Our list of instructions is now complete. We can work down all columns except Nan's. With Nan, we can, if we like, work back up the column step by step. Or we can use the row that has 4y + 4 at the beginning. "4 bags and 4 counters make 8 counters." Each bag must hold 1 counter only. Nan's number was 1. We work down Nan's column to check this.

Notes to accompany *Work Text* page 112

This page is most effective if students actually make "machines" for different rules. A thumbtack, a pointer and a piece of paper or cardboard are the only materials needed. As the pointer swings from one position to another, the students are helped to see that x is not any particular number. The machines convey this better than any verbal explanation.

In B, the rule y = 2x is given. The students have only to fill in the numbers of the 2 times table.

In C, the students may guess 7 by looking at the sequence, 4, 5, 6, . . . , 8. Or they may spot the rule: 6 goes to 4, 7 goes to 5, 8 goes to 6. We are subtracting 2; so 9 goes to 7. Then y = x − 2, merely says in shorthand that we are subtracting 2 each time. In D, also, students may guess the missing number in either of the same two ways.

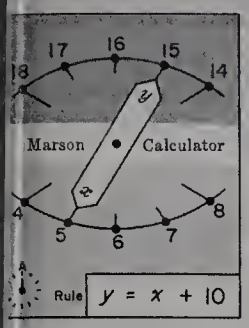
In E, the students can make up any rule they like, and work out the numbers. The rule y = x − 5 is only a suggestion. (It could lead to 3, 4, 5, 6, 7 for y.)

In F, y = 2x + ____. This means the rule is "twice the number and add ____." Students have to fill the space so that they get:

Twice 1 and add ____ gives 12
Twice 2 and add ____ gives 14

The missing number is 10. So y = 2x + 10 is the rule, and it should be used to fill in the remaining numbers.

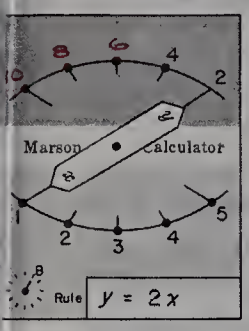
One day, Mr. Wilson, the
incipal, brought in another
Alec's machines.
"It's a calculator," he said.
The pointer swings on a
vot. When the end marked
points to 5, the end marked
points to 15. When x
oints to 8, y points to 18."
Bill volunteered, "It must
e an 'add-10' machine." Do
ou agree?



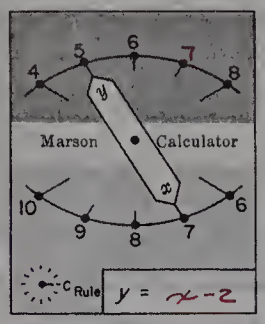
"Pointer y always points
o 10 more than pointer x.
n these instances, this rule
pplies

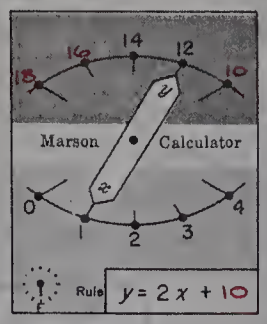
$$y = x + 10$$

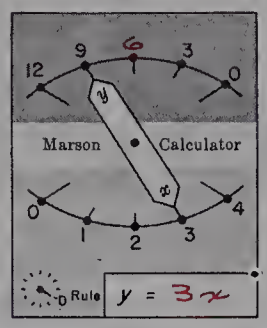
"What does the machine
elow calculate?" Mr. Wil-
on asked.

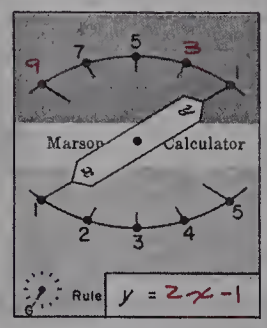


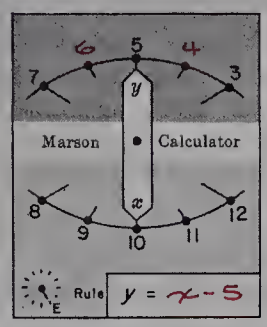
As you turn the dial in the lower left-hand corner, the rule for this calculator changes. Can you discover these rules or fill in the numbers on the calculation if anything is missing?

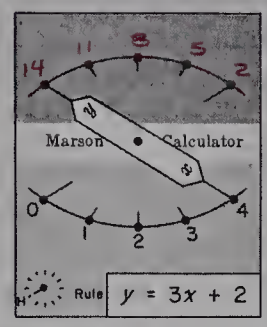












Alec has undoubtedly read Math Patterns in Science written by one of the authors, W. W. Sawyer. Alec's machine is quite similar to Professor Sawyer's machine.

(113)

The class kept a record of some of their results. They moved the pointer each time from 0 to 1 to 2 to 3, etc., in order. They noticed an interesting development no matter what rule the machine used.

They inspected the series of numbers at y. They had a common difference there.

Here are some of their records. You can complete them. (Note that C. D. is short for "common difference".)

A. Rule: $y = 2x + 1$

x	0	1	2	3	4	5	6	7
y	1	3	5	7	9	11	13	15
C.D.	2	2	2	2	2	2	2	

The difference between each successive y is the same — two in this case.

B. Rule: $y = x + 3$

x	0	1	2	3	4	5	6	7
y	3	4	5	6	7	8	9	10
C.D.	1	1	1	1	1	1	1	

The difference between 3 and 4 is 1; the difference between 4 and 5 is 1 — and so with each succeeding pair. The common difference is 1.

C. Rule: $y = 3x + 8$

x	0	1	2	3	4	5	6	7
y	8	11	14	17	20	23	26	29
C.D.	3	3	3	3	3	3	3	

Do you see any pattern to the common difference above? Can you guess in advance what the C. D. will be in the next example?

D. Rule: $y = 5x + 10$

x	0	1	2	3	4	5	6	7
y	10	15	20	25	30	35	40	45
C.D.	5	5	5	5	5	5	5	

In some of the examples to follow, it will help if you look back to page 93 in which "directed numbers" are discussed — or "positive" and "negative" numbers.

E. Rule: $y = 6x - 1$

x	0	1	2	3	4	5	6	7
y	-1	5	11	17	23	29	35	41
C.D.	6	6	6	6	6	6	6	

On a number line that extends both directions from 0, how far is it from $^-1$ to $^+5$, from $^+5$ to $^+11$, etc.?

F. Rule: $y = 9x + 3$

x	0	1	2	3	4	5	6	7
y	3	12	21	30	39	48	57	66
C.D	9	9	9	9	9	9	9	

G. Rule: $y = \frac{1}{2}x + 3$

x	0	1	2	3	4	5	6	7
y	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$
C.D	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	

Summary:

Rule A $y = 2x + 1$ → C.D. 2

Rule B $y = x + 3$ → C.D. 1

Rule C $y = 3x + 8$ → C.D. 3

Rule D $y = 5x + 10$ → C.D. 5

Rule E $y = 6x - 1$ → C.D. 6

Rule F $y = 9x + 3$ → C.D. 9

Rule G $y = \frac{1}{2}x + 3$ → C.D. $\frac{1}{2}$

Make up examples of your own to test any pattern you notice in the summary above.

Notes to accompany *Work Text* page 113

In A, the students are told the rule y = 2x + 1. They complete the table and observe how the numbers rise by steps (common difference) of 2.

In B, students observe the numbers 3, 4, 5, 6 in counting order. They fill in the remaining spaces by continuing the counting: 7, 8, 9, 10. The common difference is 1. This means there is *1 bag*, so y = 1 x + ____ or (as we usually write it) y = x + ____. It is not hard to see that we are adding 3, so y = x + 3.

In C students first have to fill the gaps: 8, . . . , 14, 17, . . . , 23 . . . , . . . This can be guessed in more than one way. One student may say, "From 14 to 17 is a step of 3, therefore the comon difference must be 3." So he starts with 8 and keeps adding 3. Another student may say, "To get from 8 to 14 in two equal steps, it is necessary to choose a number halfway from 8 to 14. The number must be 11."

Now we have y = ____x + 8. We want 2 to go to 14. Since *three* 2's plus 8 make 14, the missing number is 3.

The *Work Text* asks the student, "Do you see any pattern?" If the fact has not yet emerged in

In G, students see the numbers 1, . . . , 5, 7, . . . In all the machines on this page, the numbers rise by equal steps. So 1, 3, 5, 7, 9 is suggested as the simplest way of completing the pattern. Students may need help before they can guess the rule, y = 2x − 1; "Double and subtract 1."

In H, the rule y = 3x + 2 is given. Students merely have to apply this rule, and write in the numbers. Students may notice that the answers: 2, 5, 8, 11, 14, increase by 3 each time. This is no accident. 3x + 2 has the picture "3 bags and 2 counters." If 1 more counter is put into each bag, a total of 3 more counters has been added.

This observation provides a clue for guessing rules. Suppose a student produces the numbers:

x	0	1	2	3	4
y	3	7	11	15	19

What is his rule? The y numbers rise by steps of 4. So his picture must be 4 bags and . . . counters. The missing number is soon guessed as 3. The rule is y = 4x + 3. This idea is developed on *Work Text* page 113.

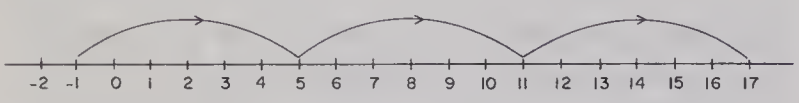
class experimentation and discussion, the students should be asked to observe the connection.

$y = 2x + 1$ has C.D. 2
$y = 3x + 8$ has C.D. 3

By observing the C.D. (common difference), it is possible to tell what number multiplies x. By observing the number that multiplies x, it is possible to predict the common difference.

So in D, if $y = 5x + \ldots$, the common difference is probably 5. Taking zero for x leads to 10 for y. So the missing number in the rule must be 10. The rule is $y = 5x + 10$. Filling in the numbers for y, students discover that the numbers do rise by steps of 5 as supposed.

In E, students are given the rule $y = 6x - 1$. They must fill in the numbers for y. They arrive at a negative number, -1, for y when they take zero for x. When they put in the differences, they will have no trouble in subtracting 5 from 11 or 11 from 17. There is a difference of 6 for all of these. *It is wise to have students do this simple subtraction first.* Now, what about the remaining subtraction? How big is the step from -1 to 5? Since 6 is the common difference for all the other steps, students will expect this step to be 6, also. Besides, this would be in agreement with the 6 in the rule, $y = 6x - 1$.

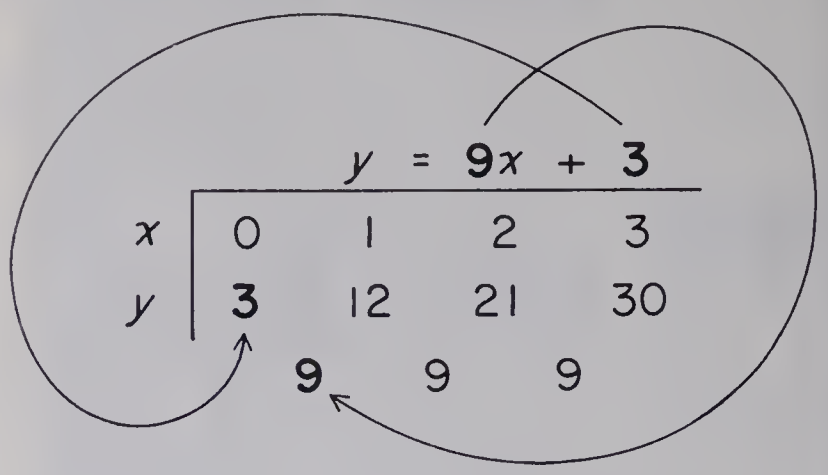


The number line clearly shows that each jump is 6. This can be verified further by examples: in money, to change a debit of \$1 to a credit of \$5, you have to pay \$6. In temperature, a change from $-1°$ to $5°$ is a rise of 6 degrees. Students will observe from all this that the presence of a negative number such as -1, does not upset the rule. (We leave negative numbers after this brief experiment.)

In F, students are given . . ., 12, . . . , 30. One argument they may use is that the number midway from 12 to 30 is 21. Another argument would be: we go from 12 to 30 in two steps. The distance is 18. Therefore, each step must be 9. We can now fill in all the numbers for y. If the common difference is 9, the rule must be $y = 9x +$ ____. It is not hard to see that $y = 9x + 3$ fits the table.

In G, students are given the rule, $y = \frac{1}{2}x + 3$. They complete the table and find (as they might have guessed) that the C.D. is $\frac{1}{2}$. The presence of fractions does not seem to spoil the connection we observed.

The patterns that students may observe and test for may be illustrated by example F.



The common difference 9 corresponds to $y = 9x + 3$. The 3 in $y = 9x + 3$ corresponds to the number for y underneath zero for x. These features will be found in every example students can compose.

Notes to accompany *Work Text* page 114

This page deals with very simple ideas. It is designed to help students become familiar with the use of shorthand. At first, the argument is presented in words, with accompanying pictures that help to illustrate the main idea. "A giant and a dwarf together fill 16 feet. If the dwarf is 3 feet high, how high is the giant?" Very few students will have difficulty with this problem. The giant is 13 feet high. Is it possible to check this answer? Yes, by adding the numbers 3 and 13: $3 + 13 = 16$.

Essentially, the problem asks students to find two numbers that add up to 16. One of these numbers refers to the dwarf's height. For short, this number is to be called d. The other number has to do with the giant's height, and is to be called g. The idea that the two numbers add up to 16 may be written $d + g = 16$.

No new thinking is used here; this is simply a short way of writing the agreement already used to solve the problem.

Full statement
A giant and a dwarf together fill 16 feet.
The dwarf is 3 feet high.
So the giant must be 13 feet high.

Shorthand
$d + g = 16$
$d = 3$
$\therefore g = 13$

The sign ∴ for "therefore" is a useful one for the students to know. It can also be read as "so it must be."

It is wise at this stage to *use letters as abbreviations* for numbers only. Thus g is *not* short for "the giant," nor even for "the giant's height." It is short for the *number* of feet in the giant's height. In the statement, "The giant is . . . feet high," g is short for the missing *number*. In the same way, d is short for the missing *number* in the sentence, "The dwarf is . . . feet high." So we do *not* say "d = the dwarf." We do not say "$d = 3$ feet." We say "$d = 3$"—the missing number (d) is 3.

It is very tedious to keep saying "d stands for the *number* of feet in the dwarf's height, whatever that may be." This is a cumbrous way to express a simple idea. But if a student writes $d = 3$ *feet*, we should tell him to erase "feet."

When solving a problem, it is usually convenient to think in terms of persons and lengths. We have done this: "A *giant* and a *dwarf* together fill 16 *feet*." If (as in our example) the giant is 13 feet high and the dwarf is 3 feet high, then $3 + 13 = 16$. This statement, $3 + 13 = 16$, is a statement about *numbers*, not about persons or lengths. The "missing numbers," d and g, add up to 16. So we can write $d + g = 16$. In an even more elementary example: "Tom has 3 apples, Mary has 5 apples. How many apples have they altogether?" The student performs the addition, $3 + 4 = 7$. He does not write $3 + 4 = 7$ *apples*. Although he may go back to the problem and write correctly: "They have 7 apples."

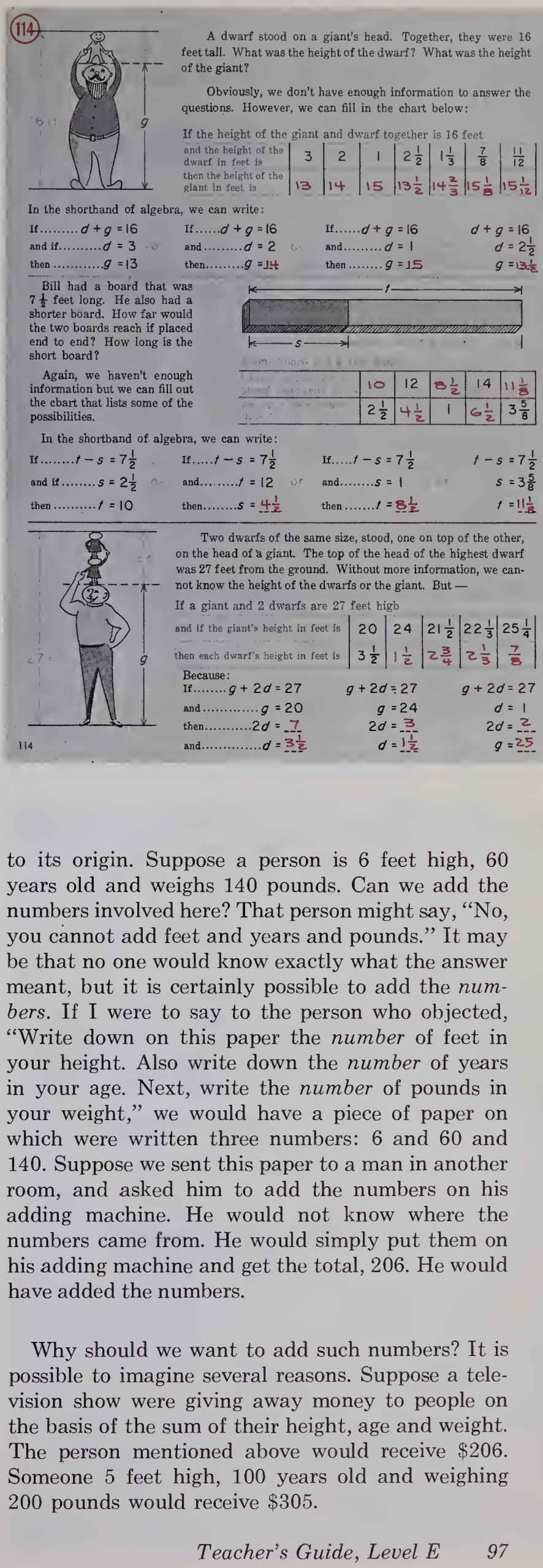
It is possible to illustrate more concisely what we mean in the following way:

How we think the problem out:
A giant and 2 dwarfs fill 27 feet.
The giant takes up 20 feet.
So 2 dwarfs must fill the other 7 feet.
So each dwarf must be 3½ feet high.

Shorthand form:
$g + 2d = 27$
$g = 20$
$\therefore 2d = 7$
$\therefore d = 3\frac{1}{2}$

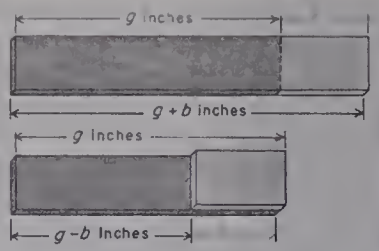
It is perhaps unfortunate that so much space has had to be devoted to what ought to be a simple matter. But there are several sources of confusion. Some engineers use letters to represent not simply numbers, but lengths, weights, etc. This is legitimate in its proper place, but confusing in introductory algebra. Also, it is easy to want to attach a number to its origin. Suppose a person is 6 feet high, 60 years old and weighs 140 pounds. Can we add the numbers involved here? That person might say, "No, you cannot add feet and years and pounds." It may be that no one would know exactly what the answer meant, but it is certainly possible to add the *numbers*. If I were to say to the person who objected, "Write down on this paper the *number* of feet in your height. Also write down the *number* of years in your age. Next, write the *number* of pounds in your weight," we would have a piece of paper on which were written three numbers: 6 and 60 and 140. Suppose we sent this paper to a man in another room, and asked him to add the numbers on his adding machine. He would not know where the numbers came from. He would simply put them on his adding machine and get the total, 206. He would have added the numbers.

Why should we want to add such numbers? It is possible to imagine several reasons. Suppose a television show were giving away money to people on the basis of the sum of their height, age and weight. The person mentioned above would receive $206. Someone 5 feet high, 100 years old and weighing 200 pounds would receive $305.


114
g
A dwarf stood on a giant's head. Together, they were 16 feet tall. What was the height of the dwarf? What was the height of the giant?
Obviously, we don't have enough information to answer the questions. However, we can fill in the chart below:
If the height of the giant and dwarf together is 16 feet
and the height of the dwarf in feet is | 3 | 2 | 1 | 2½ | 1⅓ | ⅞ | 11/12
then the height of the giant in feet is | 13 | 14 | 15 | 13½ | 14⅔ | 15⅛ | 15 1/12
In the shorthand of algebra, we can write:
If ..... d + g = 16 and if ..... d = 3 then ..... g = 13
If ..... d + g = 16 and ..... d = 2 then ..... g = 14
If ..... d + g = 16 and ..... d = 1 then ..... g = 15
d + g = 16 d = 2½ g = 13½
Bill had a board that was 7½ feet long. He also had a shorter board. How far would the two boards reach if placed end to end? How long is the short board?
t
s
Again, we haven't enough information but we can fill out the chart that lists some of the possibilities.
10 | 12 | 8½ | 14 | 11⅛
2½ | 4½ | 1 | 6½ | 3⅝
In the shortband of algebra, we can write:
If ..... t − s = 7½ and if ..... s = 2½ then ..... t = 10
If ..... t − s = 7½ and ..... t = 12 then ..... s = 4½
If ..... t − s = 7½ and ..... s = 1 then ..... t = 8½
t − s = 7½ s = 3⅝ t = 11⅛
Two dwarfs of the same size, stood, one on top of the other, on the head of a giant. The top of the head of the highest dwarf was 27 feet from the ground. Without more information, we cannot know the height of the dwarfs or the giant. But —
If a giant and 2 dwarfs are 27 feet high
and if the giant's height in feet is | 20 | 24 | 21½ | 22⅓ | 25¼
then each dwarf's height in feet is | 3½ | 1½ | 2¾ | 2⅓ | ⅞
Because:
If ..... g + 2d = 27 and ..... g = 20 then ..... 2d = 7 and ..... d = 3½
g + 2d = 27 g = 24 2d = 3 d = 1½
g + 2d = 27 d = 1 2d = 2 g = 25
114

(115) Mary folded a long strip of paper and colored the region on one side of the fold green — on the other side, blue. She measured the length of the strip before it was folded — (length of green plus length of blue.) After it was folded, she measured the part of green that was not covered by the blue.

She did this with many such strips and kept a record. Can you find the length of blue and green in each example?



(All numbers indicate number of inches.)

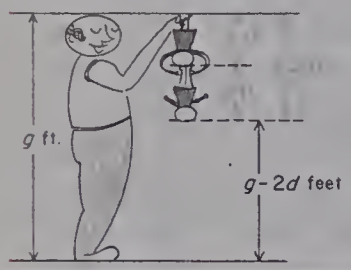
Length of green and blue together $(g+b)$	13	20	4	$15\frac{1}{2}$	$9\frac{1}{4}$	25	7	30
Length of green not covered by blue $(g-b)$	7	10	2	$4\frac{1}{2}$	$4\frac{3}{4}$	11	4	12
Length of green (g)	10	15	3	10	7	18	$5\frac{1}{2}$	21
Length of blue (b)	3	5	1	$5\frac{1}{2}$	$2\frac{1}{4}$	7	$1\frac{1}{2}$	9

If .... $g+b=13$ and .. $g-b=7$ then ..... $g=10$ and ..... $b=3$

If .... $g+b=20$ and ... $g-b=10$ then ..... $b=5$ and ..... $g=15$

If ..... $g=18$ and ..... $b=7$ then.. $g+b=25$ and .. $g-b=11$

If ..... $g=21$ and .. $g-b=12$ then ..... $b=9$ and .. $g+b=30$



A giant touches the ceiling with his head. He holds a dwarf's feet against the ceiling. This dwarf holds the feet of another dwarf.

The chart below indicates several possibilities. Of course, giants and dwarfs can be any height we want to imagine. All numbers are number of feet in length.

Height in feet of giant (g)	20	100	18	24	$33\frac{3}{4}$	47	$50\frac{1}{2}$	75
Height in feet of two dwarfs $(2d)$	4	3	1	2	$3\frac{3}{4}$	9	6	$\frac{1}{2}$
Height of one dwarf (d)	2	$1\frac{1}{2}$	$\frac{1}{2}$	1	$1\frac{7}{8}$	$4\frac{1}{2}$	3	$\frac{1}{4}$
No. of feet from bottom dwarf's head to floor $(g-2d)$	16	97	17	22	30	38	$44\frac{1}{2}$	$74\frac{1}{2}$

If ..... $g=100$ and ..... $2d=3$ then ..... $d=1\frac{1}{2}$ and.. $g-2d=97$

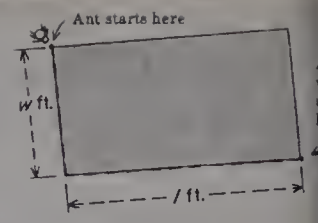
If ..... $d=1$ and .. $g-2d=22$ then ..... $2d=2$ and ..... $g=24$

If ..... $2d=9$ and.. $g-2d=38$ then ..... $g=47$ and ..... $d=4\frac{1}{2}$

If ..... $g=50\frac{1}{2}$ and.. $g-2d=44\frac{1}{2}$ then .... $2d=6$ and ..... $d=3$

(117) An ant started crawling from one corner of a rectangularly shaped room. He crawled along the wall the width (w feet) of the room plus the length (l feet) of the room. It took exactly 360 foot square tiles to cover the floor. How far did the ant crawl?

We do not have enough information, but we can work out several of the possibilities.



Area of floor is 360 square feet, or $l \cdot w = 360$.

Length of room	30	24	60	20	36	120	72	90	40	45
Width of room	12	15	6	18	10	3	5	4	9	8
Distance ant crawled	42	39	66	38	46	123	77	94	49	53

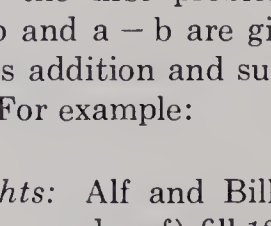
In the shorthand of algebra, we can write:

If ..... $l \cdot w = 360$ and ..... $l = 30$ then ..... $w = 12$ and ... $l + w = 42$

If ..... $l + w = 360$ and .. $l + w = 38$ then ..... $l = 20$ and ..... $w = 18$

If ..... $l + w = 360$ and ..... $w = 3$ then ..... $l = 120$ and .. $l + w = 123$

If ..... $l + w = 36$ and .../ + w = 4 then ..... l = 4 and ..... w = 9



Now suppose that the ant crawls all the way around the room — back to the corner from which he started. All we know about the size of the room is that it is rectangular in shape and covered with whole tiles that are 1-foot square.

The chart below considers some of the possibilities. All numbers refer to a number of feet or square feet.

Length of room in feet (l)	10	12	20	15	11	9	17	13
Width of room in feet (w)	15	9	17	15	7	8	13	11
Area of room in square feet $(l\,w)$	150	108	340	225	77	72	221	143
Distance ant crawls from one corner to opposite corner $(l + w)$	25	21	37	30	18	17	30	24
Twice that distance 2 $(l + w)$	50	42	74	60	36	34	60	48
Total distance ant crawls $(2\,l + 2\,w)$	50	42	74	60	36	34	60	48

Suppose the ant crawled around three of the walls and had 11 feet to go before reaching his starting place. The trip around the three walls had been 25 feet. The room must be 7 feet wide.

In the shorthand of algebra, we could write:

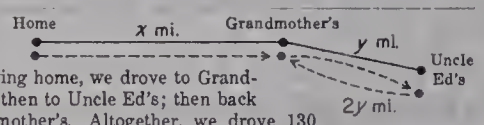
$w + l + w = 25$
$2w + l = 25$
$l = 11$
$2w = 14$
$w = 7$

(116) Headline

$a + b = 12$
$a - b = 2$
$a = 7$
$b = 5$

Mrs. Clark made two purchases. Her first purchase ($\$a$) was \$2 more than the second ($\$b$). Altogether, ($a+b$ dollars), she spent \$12. The amount of her first purchase was \$ 7 ; the second was \$ 5 .

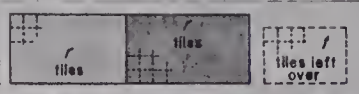
$x + 2y = 130$
$y = 20$
$2y = 40$
$x = 90$



After leaving home, we drove to Grandmother's; then to Uncle Ed's; then back to Grandmother's. Altogether, we drove 130 miles. It is 20 miles from Grandmother's to Uncle Ed's. From Grandmother's, we will have to drove 90 miles to get home.

$2r + l = 144$
$l = 24$
$2r = 120$
$r = 60$

Mr. Brown bought 144 square tiles. He had two floors the same size to cover. When he finished, there were 2 dozen tiles left over. He used 60 tiles for each floor.



$x - y = 60$
$x + 2y = 75$
$x = 65$
$2y = 10$
$y = 5$

Jack and Bill each weighed the same when school started in the fall. Jack was sick for several weeks and lost some weight, and by Thanksgiving he weighed only 60 pounds. Bill gained twice as much as Jack lost, and he weighed 75 pounds. When school started, each must have weighed 65 pounds. Jack lost 5 pounds, and Bill gained 10 pounds.

Notes to accompany *Work Text* page 115

Page 115 involves the ideas expressed on page 114. The problems are to be solved by common sense. By filling in the spaces on this page, the student gains practice in the use of mathematical shorthand.

Notes to accompany *Work Text* page 116

On this page, students are given the shorthand and asked to make up a problem that leads to the given shorthand. This approach is used in order to test the understanding students have of the basic idea.

In the first problem the shorthand notations, a + b and a − b are given. Any problem that illustrates addition and subtraction would be appropriate. For example:

Heights: Alf and Bill together (like a giant and dwarf) fill 12 feet.
Alf is 2 feet taller than Bill.

Money: Anne and Betty have 12 dollars. Anne has 2 dollars more than Betty.

Speed: A motor boat travels at 12 knots with the current, but only at 2 knots against the current. (So its speed in still water is 7 knots, and the current is 5 knots.)

Notes to accompany *Work Text* page 117

It will probably be necessary to remind students that $l \cdot w$ or lw means l times w. In the example: $l = 30$, $w = 12$, it is understood that the room contains 30 rows of 12 tiles, or 30 times 12 in all. The general rule is l times w: the number of tiles in the length of the room multiplied by the number in the width.

Work Text page 117 requires some systematic guessing. In the first table, the fourth column shows that the ant crawled 38 feet. So we want l and w to add up to 38. Since there are 360 tiles in the room, l times w must be 360. We may not find it easy to guess the numbers. It pays to make a complete list of the possibilities. 360 tiles could be *1* row of 360 tiles, or 2 rows of 180, or 3 rows of 120, and so on. We can collect these in a table:

w	l	$l + w$
1 row of	360 tiles	Ant crawls 361 feet
2 rows of	180 tiles	Ant crawls 182 feet
3	120	123
4	90	94
5	72	77
6	60	66
8	45	53
9	40	49
10	36	46
12	30	42
15	24	39
18	20	38

We stop with 18 rows of 20 tiles, since the width is understood to be less than the length. The next entry would be 20 rows of 18 tiles. All numbers that would lead to fractions have been omitted because the problem makes no mention of cutting tiles up.

It is the very last entry in the table that gives the sum we want for column four! Using the same table, it is possible to fill in all the blanks in the first problem.

For the problems on the lower half of the page, the room no longer is considered as having 360 tiles. In the first column, students are given both l and w. In the second column: w is 9, and $l + w$ has to be 21. So l must be 12, the number which, added to 9, gives 21.

In the third column, there are 340 tiles, and the room is 20 tiles long. So there must be 17 rows. The room must be 17 feet wide.

The fourth column calls for some guessing or making of tables. The distance all the way around the room is 60 feet. From one corner to the opposite (around the edge) is just half this, 30 feet. So we write 30 in the row for $l + w$. Now we are looking for two numbers, l and w, which add up to 30 and which, if multiplied together, give 225. Consider 1 and 225; 3 and 75; 5 and 45; 9 and 25; 15 and 15. The last one does the trick.

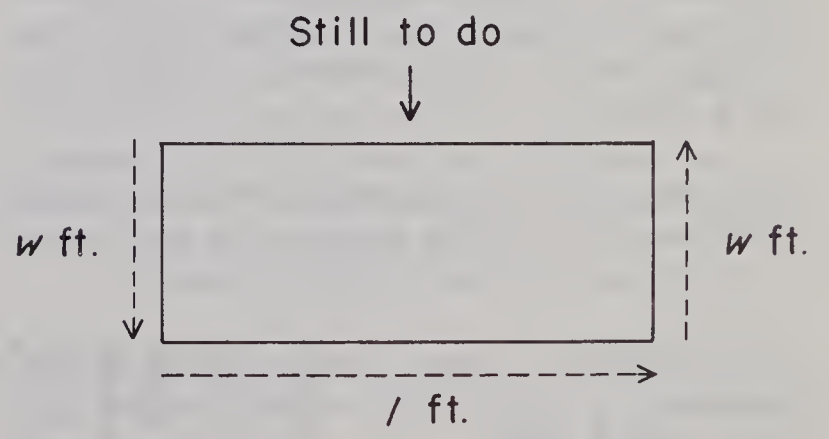
In the fifth column, a mathematician might say that two solutions were possible. The room could be 77×1 or 11×7. Logically, we must admit both solutions as correct. In practice, a room 1 foot wide is unheard of, and the only reasonable solution is $l = 11, w = 7$.

In the sixth column, it is possible to make a table showing all possible ways of getting 72 tiles: 1 row of 72; 2 of 36; 3 of 24, and so on. It is, however, not too hard to guess that 9 rows of 8 is what we want.

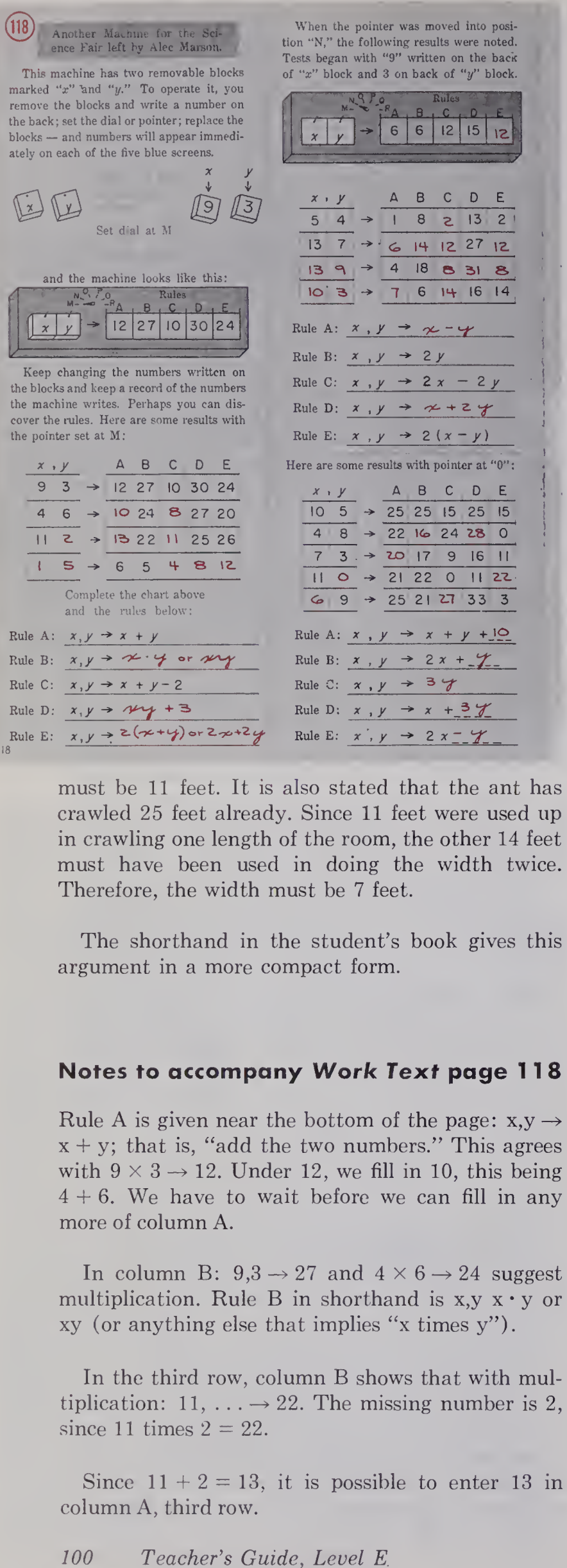
In the seventh column, 221 is a number with few factors. In fact, 17×13 and 221×1 are the only possibilities. It is easy to see which one we want.

The eighth column resembles the fifth: $143 = 13 \times 11$. Logically, we have to admit that the room could be either 13×11 or 143×1. The second solution is unrealistic.

Problem at bottom of page



When the ant has crawled around three sides, it has only one side (or length) to go. It is stated that the ant still has 11 feet to go. So the length of the room



118

Another Machine for the Science Fair left by Alec Marson.

This machine has two removable blocks marked "*x*" and "*y*." To operate it, you remove the blocks and write a number on the back; set the dial or pointer; replace the blocks — and numbers will appear immediately on each of the five blue screens.

x y

x y

9 3

Set dial at M

and the machine looks like this:

N O P Q M R

Rules

x	y	A	B	C	D	E
		12	27	10	30	24

Keep changing the numbers written on the blocks and keep a record of the numbers the machine writes. Perhaps you can discover the rules. Here are some results with the pointer set at M:

x , y	A	B	C	D	E
9 3 →	12	27	10	30	24
4 6 →	10	24	8	27	20
11 2 →	13	22	11	25	26
1 5 →	6	5	4	8	12

Complete the chart above and the rules below:

Rule A: x,y → x + y

Rule B: x,y → x · y or xy

Rule C: x,y → x + y − 2

Rule D: x,y → xy + 3

Rule E: x,y → 2(x+y) or 2x+2y

When the pointer was moved into position "N," the following results were noted. Tests began with "9" written on the back of "*x*" block and 3 on back of "*y*" block.

N O P Q M R

Rules

x	y	A	B	C	D	E
		6	6	12	15	12

x , y	A	B	C	D	E
5 4 →	1	8	2	13	2
13 7 →	6	14	12	27	12
13 9 →	4	18	8	31	8
10 3 →	7	6	14	16	14

Rule A: x , y → x − y

Rule B: x , y → 2 y

Rule C: x , y → 2 x − 2 y

Rule D: x , y → x + 2 y

Rule E: x , y → 2 (x − y)

Here are some results with pointer at "0":

x , y	A	B	C	D	E
10 5 →	25	25	15	25	15
4 8 →	22	16	24	28	0
7 3 →	20	17	9	16	11
11 0 →	21	22	0	11	22
6 9 →	25	21	27	33	3

Rule A: x , y → x + y + 10

Rule B: x , y → 2 x + y

Rule C: x , y → 3 y

Rule D: x , y → x + 3 y

Rule E: x , y → 2 x − y

must be 11 feet. It is also stated that the ant has crawled 25 feet already. Since 11 feet were used up in crawling one length of the room, the other 14 feet must have been used in doing the width twice. Therefore, the width must be 7 feet.

The shorthand in the student's book gives this argument in a more compact form.

Notes to accompany *Work Text* page 118

Rule A is given near the bottom of the page: x,y → x + y; that is, "add the two numbers." This agrees with 9 × 3 → 12. Under 12, we fill in 10, this being 4 + 6. We have to wait before we can fill in any more of column A.

In column B: 9,3 → 27 and 4 × 6 → 24 suggest multiplication. Rule B in shorthand is x,y x · y or xy (or anything else that implies "x times y").

In the third row, column B shows that with multiplication: 11, . . . → 22. The missing number is 2, since 11 times 2 = 22.

Since 11 + 2 = 13, it is possible to enter 13 in column A, third row.

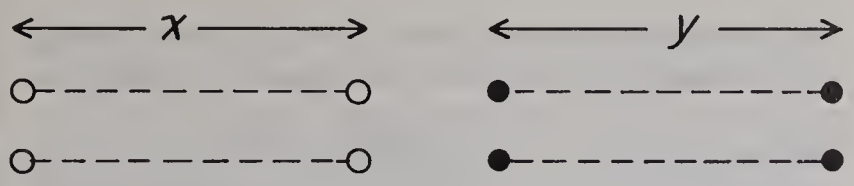
In the fourth row, the number 6 appears in column A, and the number 5 appears in column B. This means that 6 is the sum of the missing numbers, and 5 is their product. Using whole numbers, there is only way to factor 5: 5 × 1. And since 5 + 1 = 6, these must be the missing numbers. Students may ask, "Does it matter whether we write 1,5 or 5,1 in the spaces under x,y?" *In this puzzle* it does not matter, because only the operations of addition and multiplication are involved, and for these operations it does not matter which number comes first. With other puzzles on this page that involve subtraction or such an operation as "the first number added to twice the second number," it *will* matter which number is written first.

Rule C is given near the bottom of the page. x,y → x + y − 2; that is, add the numbers and subtract 2. The quickest way to do this is to take 2 from each number in column A. Also, this provides a clue which is needed in order to guess the rule for column D. Since column C has something to do with column A, perhaps column D has something to do with one of the other columns. Yes, it has. The numbers 30, 27, 25 are 3 more than 27, 24, 22 in column B. So we put 8 in the empty space. The rule for D is: "add 3 to the number in column B." But "the number in column B" already has been stated in shorthand: x · y; so rule D can be stated, "add 3 to x · y." In shorthand this would be written: 3 + x · y or 3 + xy. Rule D could also be stated, "multiply the numbers and add 3." In this case the shorthand would be: x · y + 3, or xy + 3.

What about column E? If students' minds have become fixed on addition and subtraction they may start comparing column E with the other columns in the following ways: Are the numbers in column E 6 less than those in *D*? No. Perhaps they are 14 more than those in C? No. Three less than those in B? No. 12 more than those in A? No. Students will eventually discover that the numbers in column E are not found by adding or subtracting a fixed number to the row of numbers in another column. This conclusion will force students to look for another approach. They might wonder if E could be related to another column *in some other way*, and so ask themselves: what is there in arithmetic besides adding and subtracting? Or, they might observe by comparing column E with each other column that in one instance there is a definite relationship:

E	24	20	26
A	12	10	13

The rule should then be obvious: "Double the number in column A." The number in column A is x + y. Therefore the rule for column E can be written: 2 · (x + y) or 2 (x + y).



The above picture shows x + y dots, all taken twice. An equally good way to describe the number of dots is 2x + 2y. So the rule for E could be: "Double each number and add the results."

New rules

When the pointer on the machine has been turned to N, the machine will work by new rules. Students are given these rules and are asked only to interpret the shorthand.

Rule A: x,y → x − y. "The first number, take away the second."

Rule B: x,y → 2y. "Double the second number." It makes no difference what the first number is. If a sign says: "Admission—children free, adults 50 cents," and there are x children and y adults in a party, the admission costs 50y cents. It does not matter what x, the number of children, is.

Rule C: x,y → 2x − 2y. "Double the first number. Subtract twice the second number."

Rule D: x,y → x + 2y. "To the first number add twice the second."

Rule E: x,y → 2 (x − y). Since x − y is the number in column A, the rule for E might be: "double the number in column A." Without reference to column A, the rule would be: "Begin with the first number (x). Subtract the second number (y). Double the result."

When the pointer moves to 0, no rules are shown, but students are given enough clues to make it possible for them to find the missing details.

Rule A: x,y → x + y + ____. That is, "add the numbers and then add ____." In the first row, adding 10 and 5 gives 15. To get 25, 10 more must be added. The missing number therefore is 10. The second row confirms this 4 + 8 + 10 is 22. In the third row, the answer is 20: 7 + 3 + 10. In the fourth row, zero is the missing number: 11 + ____ + 10 = 21. In the fifth row: ____ 9 + 10 = 25. The missing number is 6.

In column B, x,y → 2x + ____ means "Double the first number and add something." What is the something? The first row suggests 5, but this will not work on the third, fourth and fifth rows. In these, "something" has to be 3, 0, 9 respectively. So "something" is not a fixed number. What is it? The numbers 3, 0, 9 appear in the column for y, so the "something" is the second number: x,y 2x + y. "Double the first number and add the second." Now it is possible to fill in 16 in column B, second row.

Rule C is something like rule B when the pointer was at N. It is 3 times the second number, and 3y is the equivalent shorthand. The number 27 should be written in the space.

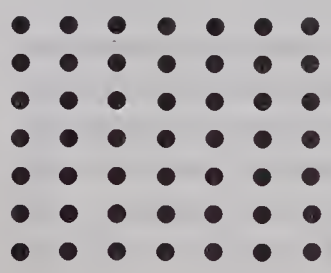
Rule D is x,y → x + ____. Again, as in rule B, something must be added to the first number. What will the something be? Taking the rows in turn, it should be: 15, ____, 9, 0, 27. Students may notice either (1) that these are the numbers in column C, or (2) that these numbers are all in the 3 times table, and are also 3 times the numbers in the y column. The number in column C is 3y, and 3y is what students should write for "something."

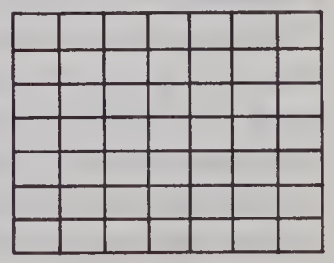
Rule E is x,y → 2x ____. This means "Double the first number and then ____." If, however, the first number is doubled, the result is always too big. 20 is more than 15; 8 is more than 0; 14 is more than 11. The next question should be, how much more? The answer for the first three rows is 5, 8, and 3. Students may notice that these are the numbers in column *y*, and these numbers must be subtracted. The rule therefore, is 2x − y, "Double the first number and subtract the second number from the result."

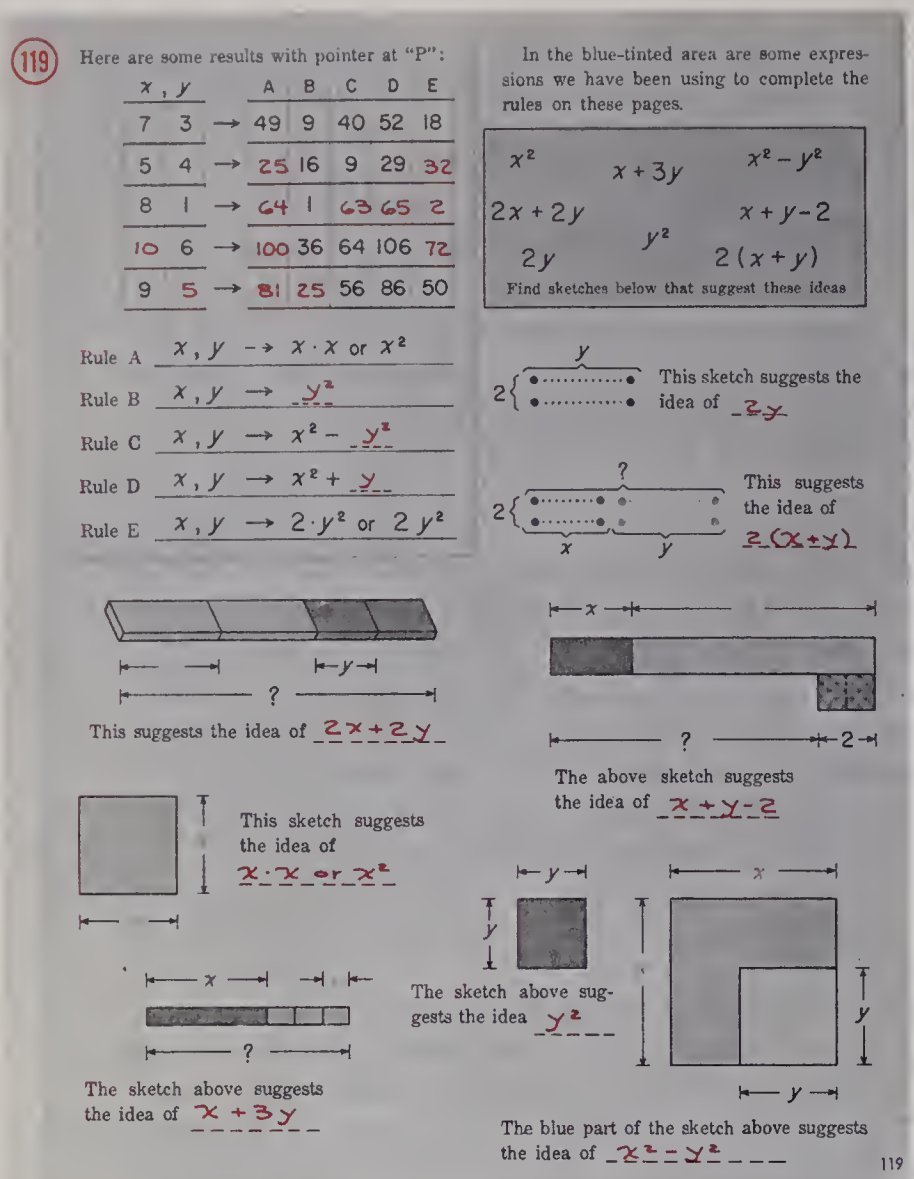
Notes to accompany *Work Text* page 119

The last set of rules on page 119 (when the pointer is set at P) is based on the idea of multiplying a number by itself: 3 times 3 or 5 times 5 or x times x. In mathematical terms this idea would be written as follows: $3 \cdot 3$ and $5 \cdot 5$ and $x \cdot x$, or 3^2 and 5^2 and x^2.

On page 119, students are told the rule for column A: $x \cdot x$; that is, the number in the x column is to be multiplied by itself. In the first row, the number is 7. So 49 should be written in column A, since 49 is 7 times 7, or 7^2. You may want to remind students that 7^2 is read "seven squared" because it can be demonstrated by the use of squares:






119
Here are some results with pointer at "P":
x, y A B C D E
7 3 → 49 9 40 52 18
5 4 → 25 16 9 29 32
8 1 → 64 1 63 65 2
10 6 → 100 36 64 106 72
9 5 → 81 25 56 86 50
Rule A x, y → x · x or x²
Rule B x, y → y²
Rule C x, y → x² − y²
Rule D x, y → x² + y
Rule E x, y → 2 · y² or 2y²
In the blue-tinted area are some expressions we have been using to complete the rules on these pages.
x² x + 3y x² − y²
2x + 2y x + y − 2
y² 2y 2(x + y)
Find sketches below that suggest these ideas
This sketch suggests the idea of 2y
This suggests the idea of 2(x + y)
This suggests the idea of 2x + 2y
The above sketch suggests the idea of x + y − 2
This sketch suggests the idea of x · x or x²
The sketch above suggests the idea y²
The sketch above suggests the idea of x + 3y
The blue part of the sketch above suggests the idea of x² − y²

this is not what is meant by $2y^2$ or 2×10^2. First, we must think of a number (y). Square it (this gives y^2). And then double the result (this gives $2 \cdot y^2$). This means that if we were to think of 10: 10×10 would give us 100, and twice that would give us 200. The answer, therefore, is 200, not 400.

It is not difficult to see how mathematicians arrived at the convention they use. In factoring such a number as 2250, the following numbers are involved: $2 \times 3 \times 3 \times 5 \times 5 \times 5$. This can be shortened to $2 \times 3^2 \times 5^3$, if it is understood that each exponent refers only to its own factor. The exponent 2 indicates 3×3. The exponent 3 indicates $5 \times 5 \times 5$. It would be erroneous to read $2 \times 3^2 \times 5^3$ as: "start with 2; multiply by 3; square the result; multiply by 5; cube the whole thing." The result would not be 2250. In fact, using *this* convention, it would be impossible to write in any way the factored form of 2250 with exponents.

The rest of the page is meant to test the student about the meaning of the algebra he has used, by asking him which picture goes with which shorthand rule. The student will find the discussion on *Work Text* pages 120-121 helpful if he has any difficulty with this page.

In column B: 9 is $3 \cdot 3$ or 3^2; 16 is $4 \cdot 4$ or 4^2; and 1 is $1 \cdot 1$ or 1^2. So $y \cdot y$ or y^2 is the shorthand for rule B.

Column C is filled in by taking the number in column A and subtracting the number in column B. since x^2 is the rule for column A, and y^2 is the rule for column B, $x^2 - y^2$ is the rule for column C. In the first row, for example, the numbers to be considered would be 7 and 3 and the procedure would be: $7 \times 7 - 3 \times 3 = 49 - 9 = 40$.

In column D, the rule is partly given: $x^2 +$ ____. "Multiply the first number by itself and add something." The "something" turns out to be 3, 4, ____, 6, 5. These are the numbers in the y column. So the rule is $x^2 + y$.

In column E, the rule is given: $2 \cdot y^2$. Since y^2 is the rule for column B, the numbers in E are twice those in B.

A probable mistake

Students are liable to make a mistake about the meaning of $2y^2$. That is, they may read the rule as: think of a number; double it; and square the result. This means that if they thought of the number 10, doubled it they would have 20 for an answer; and if then they squared 20, they would have 400. But

Two rows with *y* dots in each row helps us to think of twice *y*; that is, 2y.

There are two rows with x dots and then y dots in each row. Each row contains x + y dots. There are two rows, so the number of dots is twice (x + y) or 2 (x × y). However, a student might say he sees two sets of x dots and two sets of y dots, so he thinks of 2x + 2y. This is perfectly correct. 2 (x + y) and 2x + 2y are two different ways of writing the same thing.

The giant and dwarf picture suggest 2x − y fairly clearly; twice the height of a giant, *less* the height of a dwarf.

The two pieces of wood, each x inches long, together with two pieces of wood, each y inches long give 2x + 2y inches altogether. Here again a student may correctly point out that 2 (x + y) describes exactly the same thing.

A square with a side x inches long contains $x \cdot x$ or x^2 inches. (Compare the picture of 7^2 in these notes, a little earlier.)

A square with a side y inches long suggests y^2.

If a square with a side x inches long has a square of side y inches cut away from it, this suggests $x^2 -$

y^2. For, from an area of x^2 inches we have cut out (subtracted, taken away) an area of y^2 inches.

A stick of length x inches with 3 sticks of length y inches forms a total length of $x + 3y$ inches.

Just below the giants we see sticks of length x inches and y inches put end to end. This gives $x + y$ inches. If two inches were taken from this length, the idea would be: $x + y - 2$ inches.

Notes to accompany *Work Text* page 120

Many students will read this page without any difficulty. The purpose of the pictures is to convey the *meaning* of expressions such as $x + y$; 5a; mn; 5^2. If a student finds difficulty in understanding the use of letters, it may help to have him work gradually from familiar results in arithmetic. For example:

Five rows with 3 dots in each illustrates 5×3. What would 5 rows with 7 dots in each illustrate? 5×7. (The answer 35 is less appropriate. The picture is intended to bring out *the meaning of multiplication.*) How would you illustrate 5×8? By 5 rows with 8 dots in each. How would you illustrate 5×13? By 5 rows of 13 dots. Could we do this with 5 times any number? Yes. Alf is going to choose a number. How would you illustrate 5 times the number chosen by Alf? By 5 rows, each containing the number of dots that Alf has selected. What would be a good shorthand for "Alf's number?" We could use *a*. How would we write 5 times Alf's number? As $5 \cdot a$ or 5a. What shape would we use to picture 5a? A rectangle. How many rows? Five. How many dots in each row? Whatever number Alf chose. And the shorthand for "whatever number Alf chose" is *a*. So the rectangle has 5 rows and *a* dots in each row.

When students are ready for it, another step can be taken. We have used 5 in all our questions above. Is there something special about 5? No. We could have chosen any other number, and drawn that number of rows in the rectangle. Suppose Betty chooses how many rows there are going to be. We use *b* as shorthand for the number Betty chooses. Our rectangle now has *b* rows and *a* dots in each row. It gives us a picture of $b \cdot a$ or *ba*.

The above idea is not new; only the way of writing it is new. When we explain to children that 5 rows of 3 dots help us to imagine the meaning of 5×3, the children soon realize that 5 and 3 were chosen at random. We could use a rectangle to illustrate the multiplication of any two numbers. Indeed, this is an important point in the teaching of arithmetic. Since children may never be independently concerned with the exact numbers of some sum they did in school, we try to make it clear to them that the method employed with particular numbers can be applied to any numbers. So whenever we teach a method in arithmetic, or explain the meaning of an operation—"you write 4×3 when you have 4 groups of 3 objects"—we hope that students understand that the method, or the operation, can be applied to *any numbers*. Even in Grade 1, we have tried to convey *general ideas* about arithmetic. This means that from the beginning we have been concerned with the bases of algebra. All we are trying to do now is to give students a convenient way of writing *what they already know*.

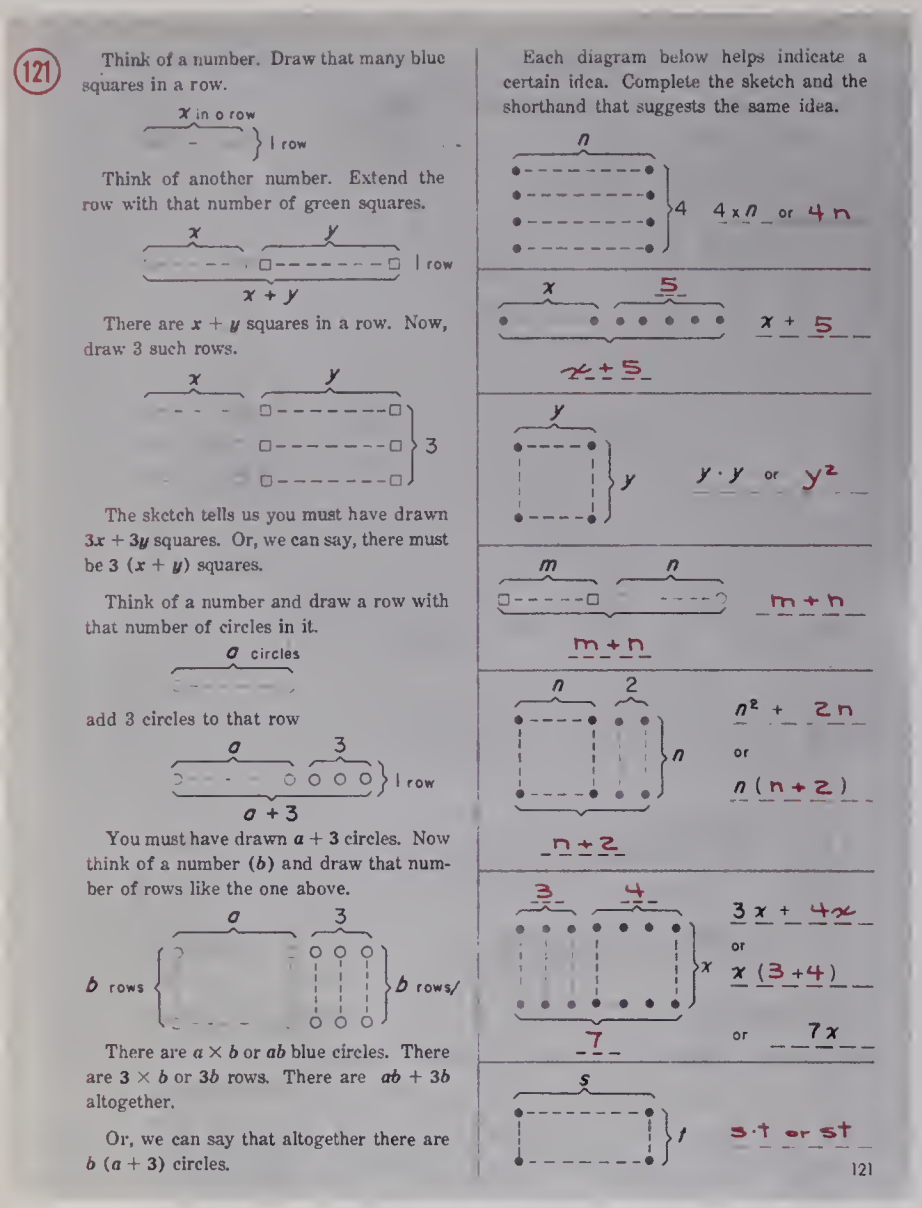
Squares

It might happen that Betty and Alf chose the same number. We would then have in our rectangle 3 rows of 3 dots, or 4 rows of 4 dots, or some similar situation. Our rectangle would be a square. This is why we call 9 "the square of 3;" 16 "the square of 4" and so on. This is simple enough. The abbreviation 3^2 for 3×3, 4^2 for 4×4 can be annoying. It takes some children years to realize that 3^2 does not mean 6. Curiously enough, they never seem to think it could mean 5. The reason is probably that we tell them 3^7 is short for $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$ because "there are seven threes there." The child hears "seven threes" and thinks immediately, "twenty-one!" It might be better to explain "3^7 means that you start with 1 and multiply by 3 seven times." Give practice in the direction they seem able to follow with the best results. What are the factors of 81? $(3 \times 3 \times 3 \times 3.)$ What is the short way to write this? Most children write 3^4 without difficulty. In the same way, $9 = 3 \times 3 = 3^2$ is usually done correctly. Students will certainly need to be reminded, fairly often, that x^2 is not just another way of writing 2^x. It may be helpful to have students make a table of squares, like the one below. They should *draw* the pictures which are merely indicated here:

Words	*Picture*	*No. of dots*	*Factors*	*Shorthand*
3 squared	3 rows of 3 dots	9	3×3	3^2
4 squared	4 rows of 4 dots	16	4×4	4^2
5 squared	5 rows of 5 dots	25	5×5	5^2

Notes to accompany *Work Text* page 121

On this page, pictures are used to demonstrate some results in algebra. The idea $x + y$ is pictured with x blue and y green objects. Three times $x + y$ is written $3 \cdot (x + y)$ or $3\ (x + y)$, the parentheses being necessary to indicate that what is meant is 3 times $x + y$, and not 3 times x only. But since there are 3 groups of x blue objects and 3 groups of y green


121
Think of a number. Draw that many blue squares in a row.
x in a row
1 row
Think of another number. Extend the row with that number of green squares.
x
y
1 row
x + y
There are x + y squares in a row. Now, draw 3 such rows.
x
y
3
The sketch tells us you must have drawn 3x + 3y squares. Or, we can say, there must be 3 (x + y) squares.
Think of a number and draw a row with that number of circles in it.
a circles
add 3 circles to that row
a
3
1 row
a + 3
You must have drawn a + 3 circles. Now think of a number (b) and draw that number of rows like the one above.
a
3
b rows
b rows/
There are a × b or ab blue circles. There are 3 × b or 3b rows. There are ab + 3b altogether.
Or, we can say that altogether there are b (a + 3) circles.
Each diagram below helps indicate a certain idea. Complete the sketch and the shorthand that suggests the same idea.
n
4
4 x n or 4n
x
5
x + 5
x + 5
y
y
y · y or y²
m
n
m + n
m + n
n
2
n
n² + 2n
or
n (n + 2)
n + 2
3
4
x
3x + 4x
or
x (3 + 4)
or
7x
7
s
t
s · t or st

The sixth picture can be described in several ways. It can be described as 3 columns with x in each, and 4 columns with x in each, making 3x + 4x in all. This can also be described as x rows with 3 + 4 objects in each, making x · (3 + 4). Or it can be described as 7 columns with x objects in each, making 7x; as x rows with 7 objects in each, making x7. Each of these descriptions is correct. They all represent the same number. This picture shows that 3x + 4x = x (3 + 4) = x7 = 7x.

In the last picture, s · t = st = t · s = ts. All four answers are correct.

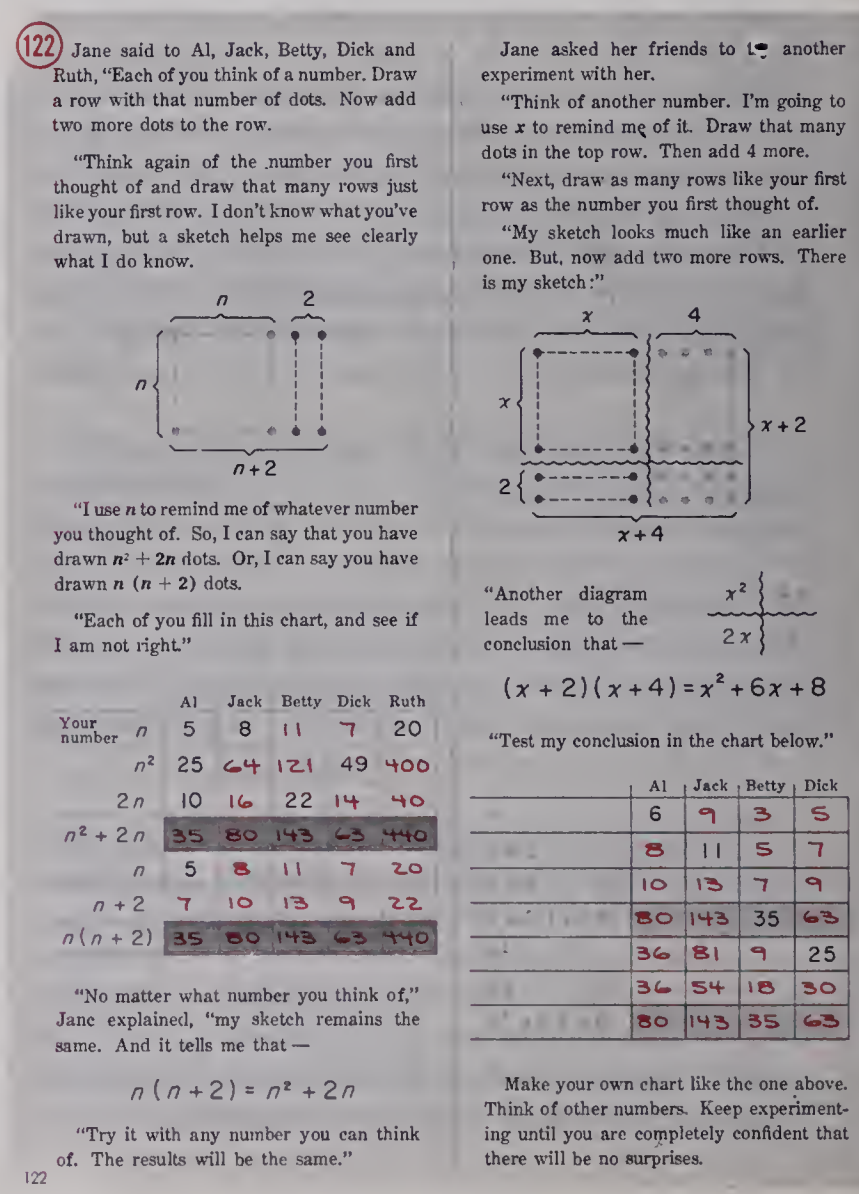
Notes to accompany *Work Text* page 122

On page 121 we had a picture that could be described either as containing $n^2 + 2n$ dots or n (n + 2) dots. Our conclusion was that for any number: $n^2 + 2n + n (n + 2)$.

Jane's picture shows that for any number $n^2 + 3n = n (n + 3)$. Following her instructions, the other children chose particular numbers and drew pictures to illustrate them. Al's picture which contains a square 5 × 5, and 3 columns with 5 dots in each shows that 5 × 8 = 5 × 5 + 3 × 5. In the table, each student works out $n^2 + 3n$ and n (n + 3) for

objects, the number of objects can be written as: 3x + 3y. Lower down a picture is used to find another way of writing b (a + 3). Note here the question of order in multiplication. Given a rectangle with *b* rows of *a* objects, students may naturally write this b · a or ba to designate "*b* times a." Or, they might describe it as *a* columns with *b* objects in each, and write *a* · *b* or *ab* or "*a* times *b*." Either form, *ba* or *ab*, is correct. Usually, because of alphabetical order, *ab* is preferred to *ba*. But *do not make an issue of this*. Mathematically, *ab* and *ba* mean exactly the same thing. Just as 3 × 4 means 12 and 4 × 3 means 12, 3 columns with b objects in each means the same as b objects in each of the 3 columns. It is purely a matter of taste whether to say "3 times b" or "b times 3." Most people prefer to write 3*b* rather than *b*3, but mathematically, *b*3 is also correct.

The first four pictures on the right side of the page involve nothing new. In the fifth picture there is a square containing n^2 objects, and two columns, each containing n objects. This could be described as representing $n^2 + 2n$ objects. It is also possible to say that there are n rows with n + 2 objects in each, so there are n · (n + 2) objects in all. It follows that $n^2 + 2n$ and n · (n + 2) are equivalent; two different ways of writing the same number.


122
Jane said to Al, Jack, Betty, Dick and Ruth, "Each of you think of a number. Draw a row with that number of dots. Now add two more dots to the row.
"Think again of the number you first thought of and draw that many rows just like your first row. I don't know what you've drawn, but a sketch helps me see clearly what I do know.
n
2
n
n+2
"I use n to remind me of whatever number you thought of. So, I can say that you have drawn n² + 2n dots. Or, I can say you have drawn n (n + 2) dots.
"Each of you fill in this chart, and see if I am not right."
Al Jack Betty Dick Ruth
Your number n 5 8 11 7 20
n² 25 64 121 49 400
2n 10 16 22 14 40
n² + 2n 35 80 143 63 440
n 5 8 11 7 20
n + 2 7 10 13 9 22
n(n + 2) 35 80 143 63 440
"No matter what number you think of," Jane explained, "my sketch remains the same. And it tells me that —
n (n + 2) = n² + 2n
"Try it with any number you can think of. The results will be the same."
Jane asked her friends to try another experiment with her.
"Think of another number. I'm going to use x to remind me of it. Draw that many dots in the top row. Then add 4 more.
"Next, draw as many rows like your first row as the number you first thought of.
"My sketch looks much like an earlier one. But, now add two more rows. There is my sketch:"
x
4
x
x + 2
2
x + 4
"Another diagram leads me to the conclusion that —
x²
2x
(x + 2)(x + 4) = x² + 6x + 8
"Test my conclusion in the chart below."
Al Jack Betty Dick
6 9 3 5
8 11 5 7
10 13 7 9
80 143 35 63
36 81 9 25
36 54 18 30
80 143 35 63
Make your own chart like the one above. Think of other numbers. Keep experimenting until you are completely confident that there will be no surprises.
122

the particular number he has chosen and finds they are the same.

On the right side of the page, Jane has drawn a picture which shows that for any number (x) you may choose, x + 2 rows with x + 4 dots in each can be broken up into the following parts: a square x by x; six lines of x dots (four upright and two lying on their sides); and 8 dots. This shows that $(x + 2) \cdot (x + 4) = x^2 + 6x + 8$. Following her instructions, Jane's friends draw their own pictures to illustrate the same thing for particular numbers of their own choosing. Since Betty has chosen 3, her picture illustrates 5×7 and it can be broken up into a square 3×3; six threes; and 8 dots. And $35 = 3^2 + 6 \cdot 3 + 8$.

In the table, students are told that Al chose 6. Jack had 11 for x + 2, so he must have started with 9. Dick had 25 for x^2, so he must have started with 5, since $5 \times 5 = 25$. Betty had 35 dots altogether in her rectangle. It could only have been 35×1 or 5×7; and 5×7 makes sense. Betty chose 3 for x, so x + 2 was 5 and x + 4 was 7.

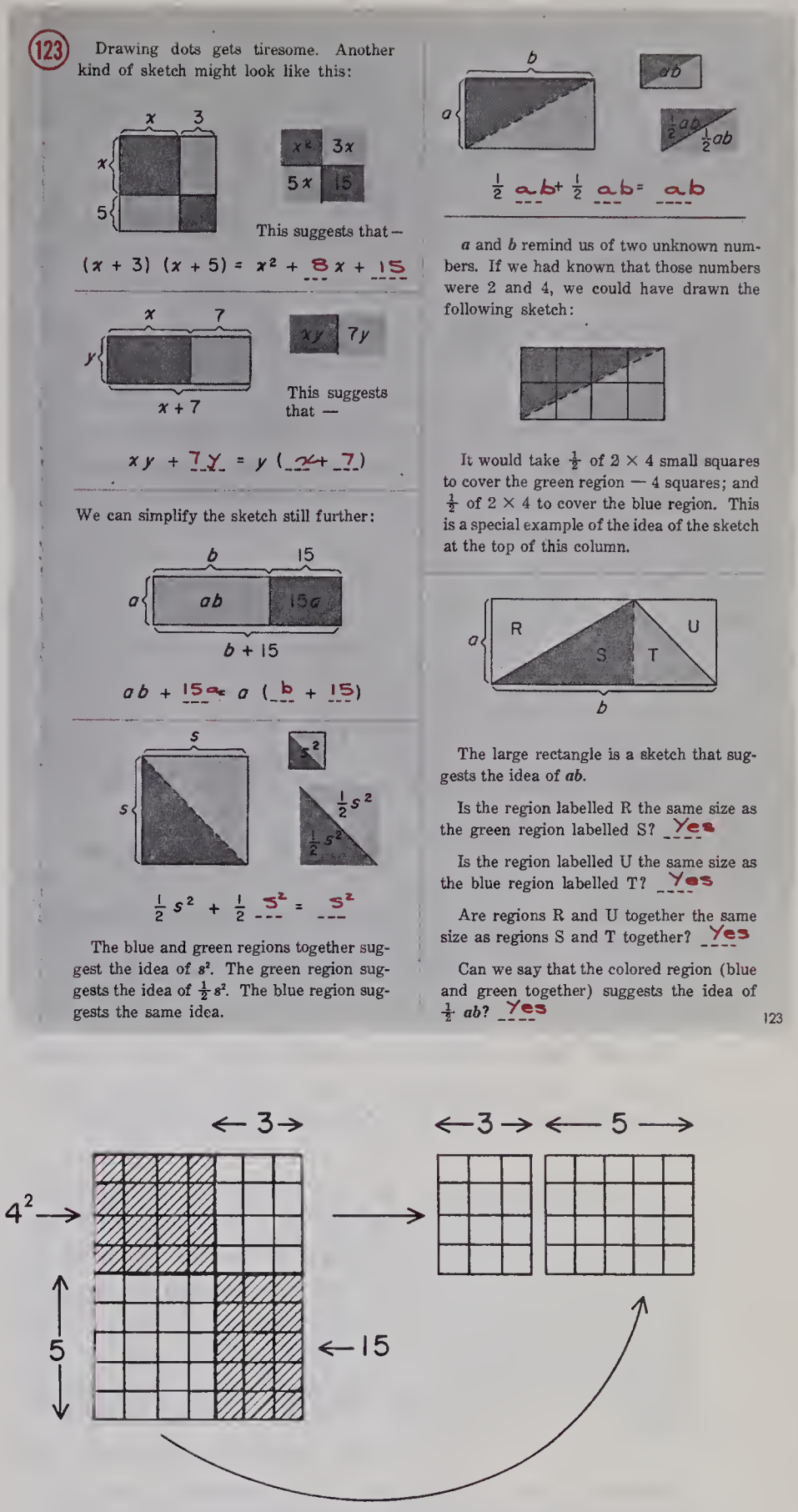
Notes to accompany *Work Text* page 123

On this page the dots have not been drawn. But if we write 3x in a rectangle it means we imagine x rows of 3 dots in each. It is also possible to think in terms of *area*, so that we can say a rectangle x inches by 3 inches is covered by 3x inch squares.

To make sure that students are not confusing area with perimeter, draw a rectangle 3 by 5 and ask for its area. Some students will probably answer 16—the distance around the edges. If so, ask them to make a number of inch squares to see how many they would need to paste over the rectangle in order to cover it. It might be helpful if the terms "area" and "pasting over" were to become closely associated in students' minds. (The idea of area is essential for the work on pages 126 and 127.)

Work Text page 123 is similar to page 122. Students are to write an expression for the number of dots in (or the area of) a complete region. Such a region is broken into parts. For example, the first rectangle is x + 5 by x + 3 inches. Its total area is $(x + 5) \cdot (x + 3)$ square inches. But the rectangle consists of four parts, with areas x^2, 3x, 5x, and 15. Adding these, we have $x^2 + 8x + 15$.

If any student questions 3x + 5x + 8, it is possible to discuss this in various ways: (1) we can go back to dots, and use the same arguments as those on page 122; (2) we can choose some number, say 4, for x and demonstrate with graph paper:


123
Drawing dots gets tiresome. Another kind of sketch might look like this:
x
3
x
5
x²
3x
5x
15
This suggests that —
(x + 3) (x + 5) = x² + 8x + 15
x
7
y
x + 7
xy
7y
This suggests that —
xy + 7y = y (x + 7)
We can simplify the sketch still further:
b
15
a
ab
15a
b + 15
ab + 15a = a (b + 15)
s
s
s²
½s²
½s²
½ s² + ½ s² = s²
The blue and green regions together suggest the idea of s². The green region suggests the idea of ½ s². The blue region suggests the same idea.
b
a
ab
½ab
½ab
½ ab + ½ ab = ab
a and b remind us of two unknown numbers. If we had known that those numbers were 2 and 4, we could have drawn the following sketch:
It would take ½ of 2 × 4 small squares to cover the green region — 4 squares; and ½ of 2 × 4 to cover the blue region. This is a special example of the idea of the sketch at the top of this column.
a
R
S
T
U
b
The large rectangle is a sketch that suggests the idea of ab.
Is the region labelled R the same size as the green region labelled S? Yes
Is the region labelled U the same size as the blue region labelled T? Yes
Are regions R and U together the same size as regions S and T together? Yes
Can we say that the colored region (blue and green together) suggests the idea of ½ ab? Yes
123

Using pieces of graph paper 3×4 and 5×4, it is possible to make a rectangle 8×4; (3) we can challenge students to try to "think of a number" x which would result in 3x + 5x being different from 8x.

The next two sketches are simple. A rectangle is divided into two pieces. The area of the whole rectangle is written. The sum of the areas of the two parts is written. There are two different ways of writing the same number.

The remaining sketches involve halving an area. These sketches lead to the conclusion that the area of a triangle is half the area of a certain rectangle. Students should not merely parrot a formula. For

each given triangle, they should sketch the rectangle of which the triangle covers one half. Using common sense they should be able to find the area of the rectangle: how many inch squares would be needed to cover it? They should then be able to see that the triangle occupies just ½ of the area of the rectangle.

The argument in the *Work Text* which is given at the end of page 123 is probably too detailed. This is the limitation of a *book*. A *teacher*, however, can observe the class. You may very well want to say to the students, "Do not read what the book says. Can you puzzle out for yourselves the area of the colored triangle marked S, T?"

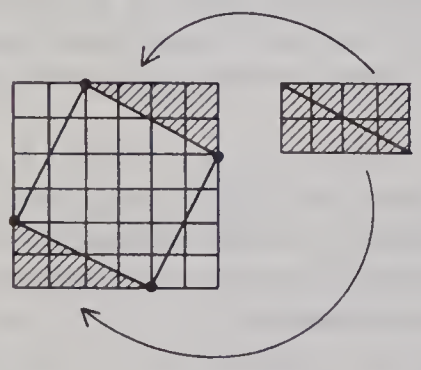
Notes to accompany *Work Text* page 124

On this page, two methods are given for finding the area of a tiled square. According to the first method, the blue square is cut into pieces which are then rearranged to cover squares on the paper exactly.

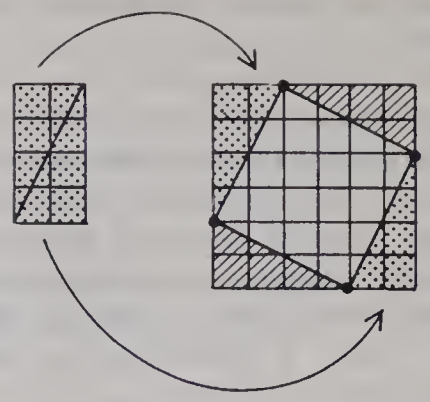
It will help students to understand the meaning of area, if you demonstrate for them by using paper or cardboard which you can cut and reassemble.

In the first method we use *addition:* we add the areas of C, D, E, F, G. In the second method we use *subtraction*. The tiled square is drawn on a piece of graph paper, 6×6. So, 36 tiles would be needed to cover it. We have to decide how many tiles would be needed in order to cover the triangles at the corners. It may help to consider these triangles in pairs.

With 8 tiles, I can cover two corners. I have now used 8 of my 36 tiles. I have 28 left.



With 8 more tiles, I cover the other two corners. This leaves me with $28 - 8 = 20$ tiles to cover the square in the middle.

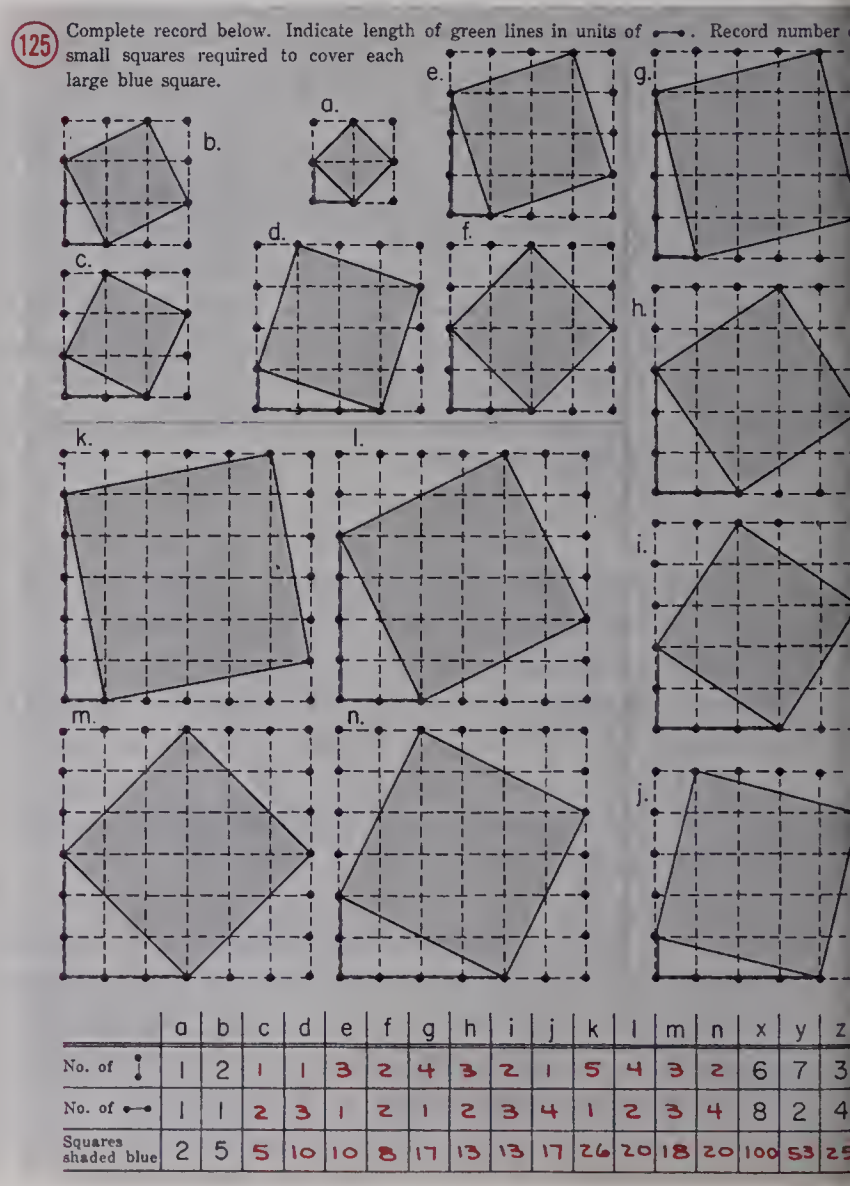


This answer will agree with the answer found by the first method.

Students should be familiar with the second method, since it will be useful later on. See, for example, the demonstration described at the end of the notes to page 126.

Notes to accompany *Work Text* page 125

This page consists of exercises that reflect the ideas presented on page 124. Students are asked to find the areas of a number of blue squares. To do so, students can use either method described on page 124, or they may have some methods of their own. So long as the correct answer is reached, any method is allowed.



	a	b	c	d	e	f	g	h	i	j	k	l	m	n	x	y	z
No. of ⁞	1	2	1	1	3	2	4	3	2	1	5	4	3	2	6	7	3
No. of ⊷	1	1	2	3	1	2	1	2	3	4	1	2	3	4	8	2	4
Squares shaded blue	2	5	5	10	10	8	17	13	13	17	26	20	18	20	100	53	25

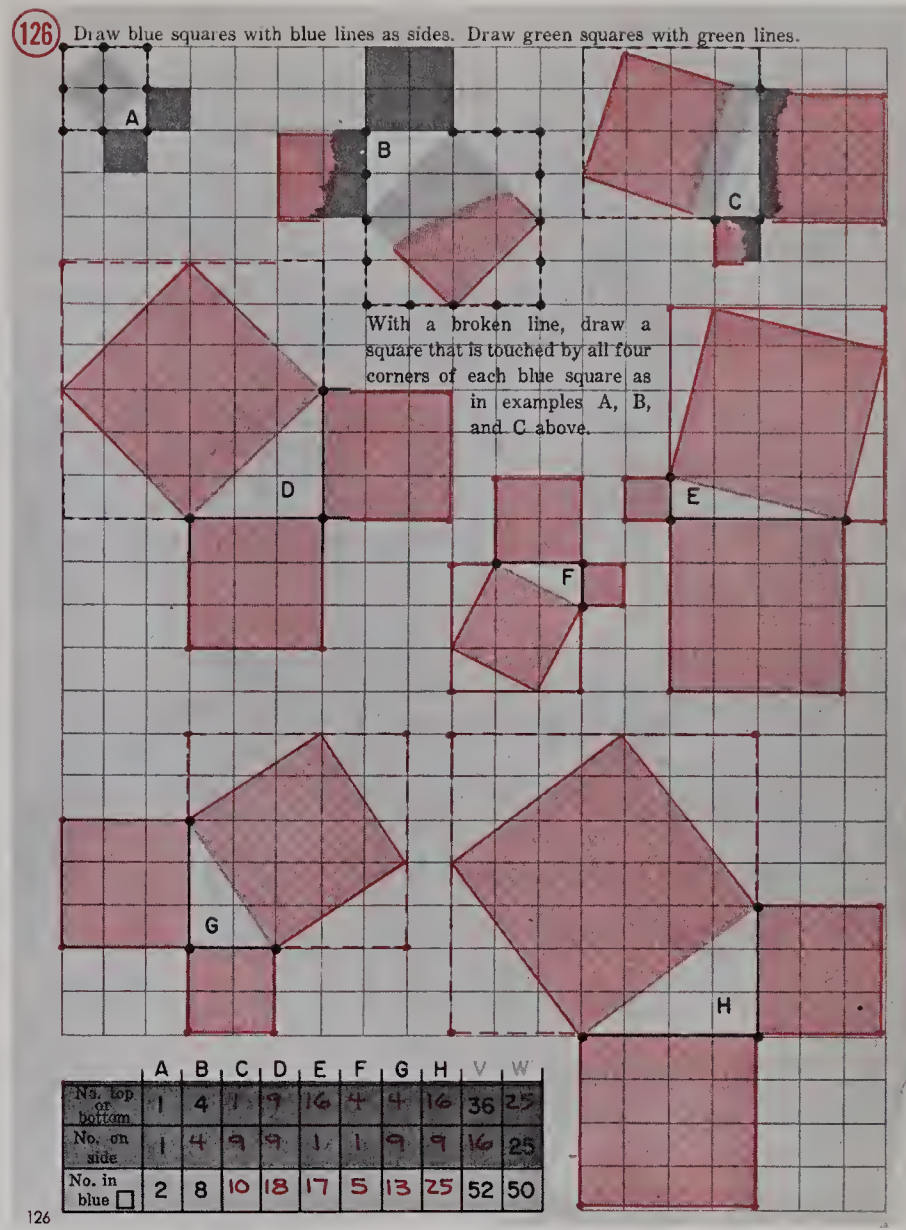
Students will have to draw cases x, y, z for themselves, since there is not enough room on the page. For any exercise that a student finds difficult, he should make his own *large-scale* drawing on squared paper, and cut it out.

Cases *x* and *z* deserve attention. In case *x*, the blue square contains 100 tiles. Could 100 tiles be arranged to make a square? In 10 rows of 10. So the side of the blue square must be 10. In case *z* the blue square takes 25 tiles. Therefore, the square is 5×5. Here is the well-known 3, 4, 5 triangle which the ancient Egyptians used for getting square corners of buildings and pyramids.

The rule that lies behind these numbers will be *discovered* by the students on *Work Text* page 126. Do not give it away by telling them what it is!

Notes to accompany *Work Text* page 126

Teachers will recognize the figure of the Pythagorean Theorem on this page. Students presumably will not, but they should be allowed to discover it for themselves.


126
Draw blue squares with blue lines as sides. Draw green squares with green lines.
A
B
C
With a broken line, draw a square that is touched by all four corners of each blue square as in examples A, B, and C above.
D
E
F
G
H
A B C D E F G H V W
Ns. top or bottom 1 4 1 9 16 4 4 16 36 25
No. on side 1 4 9 9 1 1 9 9 16 25
No. in blue □ 2 8 10 18 17 5 13 25 52 50

Students are asked first to find the area of each blue square, just as they did on the previous page. They are then asked to record their answers in the bottom row of the table. In the top two rows of the table, students are to record each green square. Thus, in figure H, there are 9 "tiles" in one green square and 16 tiles in the other. We enter 16 and 9 in column H of the table.

When students have completed the table, ask, "What do you notice about the table?" Try to allow enough time for each student to make the discovery, before having someone answer. The answer is that the numbers in the bottom row of the table can be found by adding the numbers in the first two rows. That is, there are just enough tiles in the two green squares to cover the blue square.

Ask students to work exercises following this procedure: If the two green sides are 3 and 4, what is the blue side?

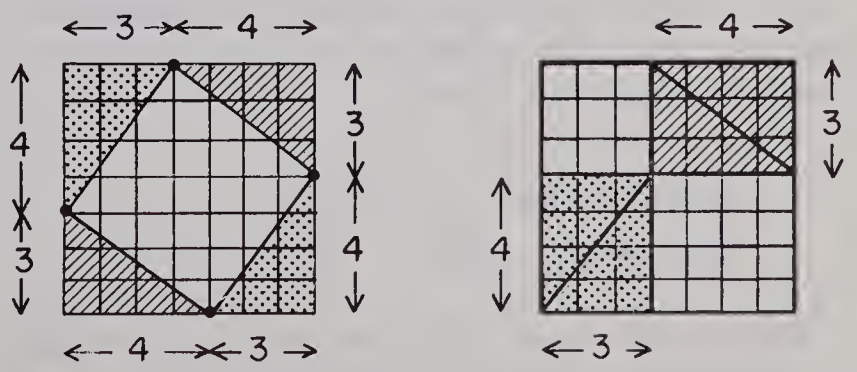
$3 \times 3 = 9$
$4 \times 4 = 16$
Add $25 = 5 \times 5$. Answer—5.

An exact answer could also be found if the green sides were 6 and 8; 5 and 12; 9 and 12; 7 and 24; 12 and 16; 9 and 40. But, if the green sides were 4 and 5, what would the blue side be?

$4 \times 4 = 16$
$5 \times 5 = 25$
Add $41 = ? \times ?$

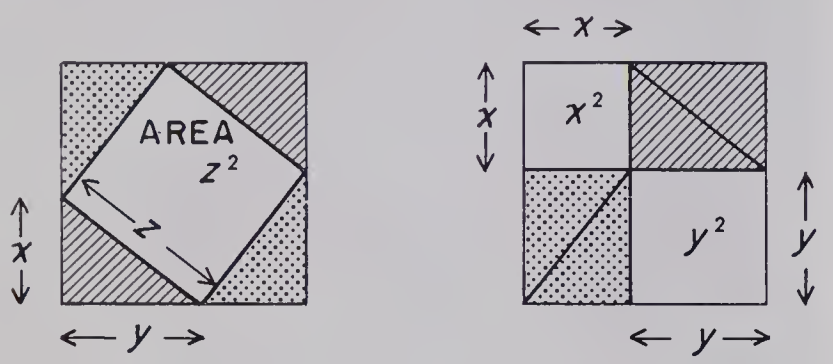
Since $6 \times 6 = 36$ and $7 \times 7 = 49$, it is obvious that 41 tiles are too much for a square 6×6, and too little for a square 7×7. The answer might be "Between 6 and 7." If students suggest $6\frac{1}{2}$, try it. $6\frac{1}{2} \times 6\frac{1}{2} = 42\frac{1}{4}$, which is more than 41, but much nearer to it than 49. "A little less than $6\frac{1}{2}$" would be a good answer. Have students measure the blue line in order to check this answer.

To show how a blue square does contain as many tiles as its companion two green squares, the following demonstration may be useful: If there are two squares, each 7×7, one of these will just cover the other, and each square will require 49 tiles.



If the four shaded triangles from the right-hand square were cut out it would be possible to place them over the shaded triangles of the left-hand square. Do this. Now the unshaded part of the right-hand square must be sufficient to cover the unshaded part of the left-hand square. But on the right there are two squares, one 3×3, the other 4×4.

There is nothing special about 3 and 4. The same demonstration could be done with *any* numbers. If we write x instead of 3, y instead of 4, and z for the side of the tiled blue square, we are able to state the idea in its general form.



Comparing areas, we have $z^2 = x^2 + y^2$, the algebraic statement of the Pythagorean Theorem. *It is necessary for students to understand this thoroughly before proceeding with page 127.*

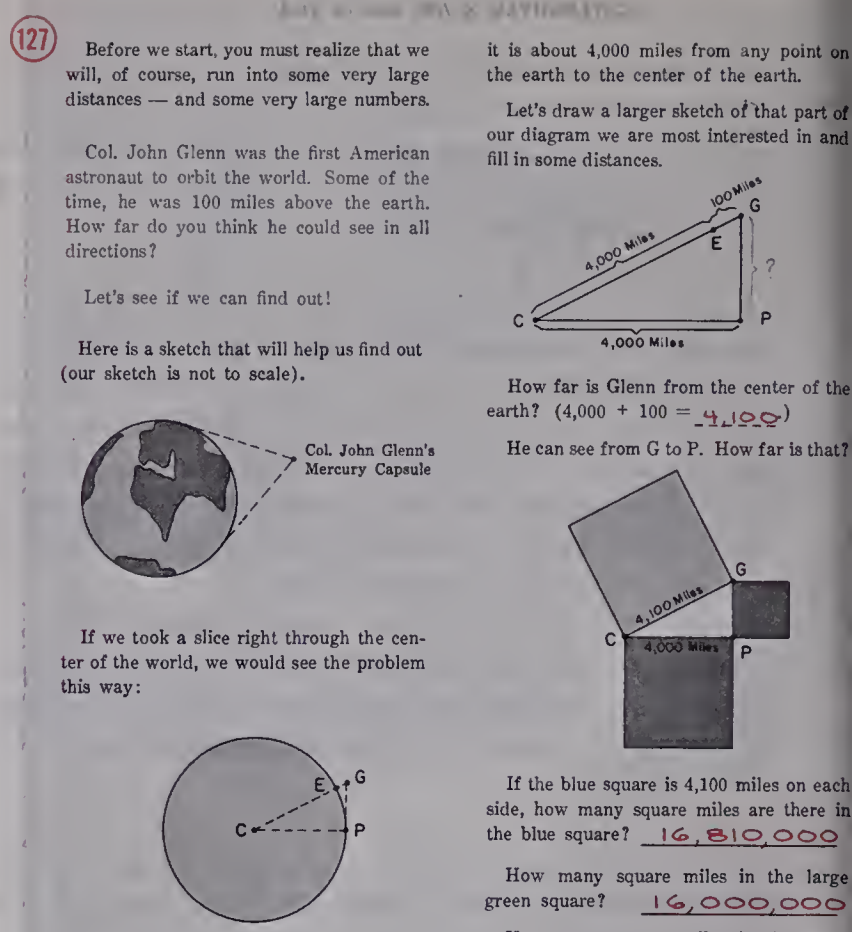
Notes to accompany *Work Text* page 127

The students may not immediately realize that P is the farthest point on the earth's surface that is visible from G. To discuss this, locate a point somewhere on the opposite side of the earth to G. Could Glenn see this point? No, there is earth between. Now, moving toward P, locate a series of points and ask students to have you stop as soon as you have reached a point that is within Glenn's sight. They should do this when you have reached point P.

The angle between CP and GP *looks* like the corner of a square. We do not try to *prove* this—that will come later, when the student begins formal geometry. For now we assume that this angle is 90°. This gives us the kind of a triangle that we have been studying in the last few *Work Text* pages. We know the blue square has the same area as the two green squares combined, so we apply the pattern we used with the 3, 4, 5 triangle.

Green squares $\begin{cases} 3 \times 3 = 9 \\ 4 \times 4 = 16 \end{cases}$
Blue square—Add $25 = 5 \times 5$

Now suppose we had this question: The blue side is 10 inches long. One green side is 8 inches long.


127
Before we start, you must realize that we will, of course, run into some very large distances — and some very large numbers.
Col. John Glenn was the first American astronaut to orbit the world. Some of the time, he was 100 miles above the earth. How far do you think he could see in all directions?
Let's see if we can find out!
Here is a sketch that will help us find out (our sketch is not to scale).
Col. John Glenn's Mercury Capsule
If we took a slice right through the center of the world, we would see the problem this way:
E
G
C
P
G suggests Glenn, and he is 100 miles above the earth at point E. From that height, he can see as far as point P on the surface of the earth. Of course, point E and point P are about the same distance from C — the center of the earth — and it is about 4,000 miles from any point on the earth to the center of the earth.
Let's draw a larger sketch of that part of our diagram we are most interested in and fill in some distances.
100 Miles
G
4,000 Miles
E
?
C
P
4,000 Miles
How far is Glenn from the center of the earth? (4,000 + 100 = 4,100)
He can see from G to P. How far is that?
G
4,100 Miles
C
4,000 Miles
P
If the blue square is 4,100 miles on each side, how many square miles are there in the blue square? 16,810,000
How many square miles in the large green square? 16,000,000
How many square miles in the small green square? 810,000
How long is each side of the small green square? 900 miles
(Find out yourself before turning the page.)
12

How long is the other green side? We write the same pattern of calculation, leaving blanks where the numbers are still unknown.

Green squares $\begin{cases} 8 \times 8 = 64 \\ \times \quad = \end{cases}$
Blue square—Add $100 = 10 \times 10$

Can we fill in the blanks? Not difficult. $64 +$ ____ $= 100$. The missing number is 36. Now we have

$8 \times 8 = 64$
$\times \quad = 36$
$100 = 10 \times 10$

We want 6×6 to finish this off. So the other green side is 6 inches long.

Exactly the same method works for Glenn's horizon. We know the blue side (4,100) and one green side (4,000). So, we write:

Green $\begin{cases} 4{,}000 \times 4{,}000 = 16{,}000{,}000 \\ \_\_\_\_ \times \_\_\_\_ = \_\_\_\_\_ \end{cases}$
Blue Add $16{,}810{,}000 = 4{,}100 \times 4{,}100$

The missing number in the addition is 810,000 which, fortunately, is exactly 900×900. The answer, therefore, is: from G to P it is 900 miles.

The same method can be used to predict how far you could see from an airplane or satellite at any height. Do not expect the arithmetic to be as simple as here! We chose our problem carefully!

For example, with a plane 1 mile up, the work would go like this:

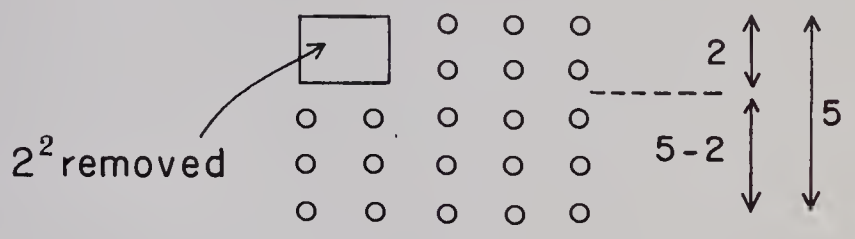
$$\underline{\quad} \times \underline{\quad} = \underline{\quad}$$
$$4{,}000 \times 4{,}000 = 16{,}000{,}000$$
$$\text{Add} \qquad 8{,}001$$
$$16{,}008{,}001 = 4{,}001 \times 4{,}001$$

Now 90×90 is 8,100, too big for our purpose. Also $89 \times 89 = 7{,}921$, is too small. So the answer comes "more than 89, less than 90 miles." This is near enough for a passenger in an airplane.

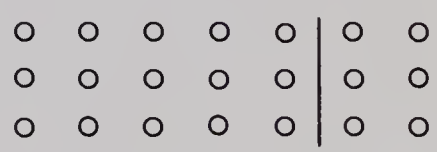
Notes to accompany *Work Text* page 128

On this page, Alec Marson shows his friends how to solve the problem about Glenn. The reasoning behind his calculations will be understood from the notes already given for page 127. Alec later shows how to shorten the arithmetic involved. First he found 810,000 by subtracting $4{,}000^2$ from $4{,}100^2$. In other words, his first method (and the method we used earlier) requires us first to calculate $4{,}100^2 - 4{,}000^2$. But Alec has hit upon a formula: "the difference of the squares equals the sum times the difference of the numbers" or, as Alec gives it in symbols, $x^2 - y^2 = (x + y) \cdot (x - y)$. Taking 4,100 for x and 4,000 for y, $x + y$ becomes 8,100, and $x - y$ simply 100. So $(x + y) \cdot (x - y)$ is $8{,}100 \times 100 = 810{,}000$. This method saves us the trouble of squaring 4,100 and 4,000. As before, we have to finish by seeing that $900 \times 900 = 81{,}000$.

Alec ends by giving some pictures to show that he has not simply memorized $x^2 - y^2 = (x + y)(x - y)$. A student should *remember* that *there is a rule* for factoring the *difference of squares*. But, to avoid talking nonsense, he should *think out for himself* what the rule is. He may start with a special case, say $5^2 - 2^2$. Here is a picture of 5^2 with 2^2 taken away:



What remains can be rearranged as a rectangle, 3×7:



One side of the rectangle is 7, which is $5 + 2$. The other side is 3, which is $5 - 2$. So $5^2 - 2^2$ equals $5 + 2$ times $5 - 2$. The same rearrangement is possible with any two numbers.

Teacher Observations and Ideas:

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